Practical - 02 objectives

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1. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has
   (a) 3 subgroups of order 2. ✓(b) 7 subgroup of order 2 (c) 6 subgroups of order 2.
   (d) 9 subgroups of order 2.

Solu:- $Z_2 = \{0, 1\}$

$Z_2 \times Z_2 \times Z_2 = \{(0,0,0), (0,1,0), (0,0,1), (1,0,0), (1,1,0), (0,1,1), (1,0,1), (1,1,1)\}$

$o(a,b,c) = \text{lcm}\{o(a), o(b), o(c)\}$

$o(0) = 1, \ o(1) = 2$ in Z_2

$H_1 = \{(0,0,0), (0,1,0)\}$, $(0,1,0) + (0,1,0) = (0,0,0)$

$H_2 = \{(0,0,0), (0,0,1)\}$, $(0,0,1) + (0,0,1) = (0,0,0)$

$H_3 = \{(0,0,0), (1,0,0)\}$, $(1,0,0) + (1,0,0) = (0,0,0)$

$H_4 = \{(0,0,0), (1,1,0)\}$, $(1,1,0) + (1,1,0) = (0,0,0)$

$H_5 = \{(0,0,0), (1,0,1)\}$

$H_6 = \{(0,0,0), (0,1,1)\}$

$H_7 = \{(0,0,0), (1,1,1)\}$

2. The order of any non-identity element in $\mathbb{Z}_3 \times \mathbb{Z}_3$ is
   ✓(a) 3 (b) 9 (c) 6 (d) none of these.

$Z_3 = \{0, 1, 2\}$

$Z_3 \times Z_3 = \{(0,0), \underline{(0,1)}, (0,2), (1,0), (2,0), (1,1), (2,2), (2,1), (1,2)\}$

$(0,1) + (0,1) + (0,1) = (0,0)$

$O((0,1)) = 3$

3. Which of the following statements is false?
   (a) $\mathbb{Z}_3 \times \mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_{15}$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$
   ✓(c) $\mathbb{Z}_9 \times \mathbb{Z}_9$ is isomorphic to $\mathbb{Z}_{27}$ (d) $\mathbb{Z}_4 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_{12}$

$O(Z_3 \times Z_5) = 3 \times 5 = 15$ as $(3,5) = 1$, $Z_3 \times Z_5 \cong Z_{15}$

$O(Z_2 \times Z_3) = 6$, $O(Z_6) = 6$, $Z_2 \times Z_3 \cong Z_6$

$O(Z_9 \times Z_9) = 81$, $Z_9 \times Z_9 \ncong Z_{27}$

$O(\mathbb{Z}_9 \times \mathbb{Z}_9) = 81$, $\mathbb{Z}_9 \times \mathbb{Z}_9 \ncong \mathbb{Z}_{27}$

$O(\mathbb{Z}_4 \times \mathbb{Z}_3) = 12$, $\mathbb{Z}_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_{12}$

4. The group $S_3 \times \mathbb{Z}_2$ is isomorphic to
   (a) $\mathbb{Z}_{12}$ (b) A_4 (c) D_6 (d) $\mathbb{Z}_6 \times \mathbb{Z}_2$

$O(S_3) = 6$, $O(\mathbb{Z}_2) = 2$

$O(A_4) = 12$, $S_3 \times \mathbb{Z}_2 \cong A_4$

5. Let $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_{15}$ and $G_2 = \mathbb{Z}_6 \times \mathbb{Z}_{10}$, then
   (a) G_1 and G_2 are cyclic groups of order 60.
   (b) G_1 and G_2 are not cyclic groups.
   (c) G_1 is cyclic but G_2 is not cyclic group.
   (d) G_1 is not cyclic but G_2 is a cyclic group.

In $\mathbb{Z}_4$, $O(1) = 4$, In $\mathbb{Z}_{15}$, $O(1) = 15$

$O(1,1)$ in $\mathbb{Z}_4 \times \mathbb{Z}_{15} = 60$

$\Rightarrow G_1$ is cyclic

In $\mathbb{Z}_6$, $O(1) = 6$, in $\mathbb{Z}_{10}$, $O(1) = 10$

$O(1,1)$ in $\mathbb{Z}_6 \times \mathbb{Z}_{10} = \text{lcm}\{6, 10\} = 30$

2	6, 10
3	3, 5
5	1, 5
	1, 1

$\therefore G_2$ is non cyclic

6. . Which is true about groups?
   (a) $\mathbb{Z}_4 \times \mathbb{Z}_2$ is isomorphic to $V_4 \times \mathbb{Z}_2$.
   (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $V_4 \times \mathbb{Z}_2$.
   (c) $V_4 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.
   (d) D_4 (the dihedral group of order 8) is isomorphic to Quaternion group Q_8 of order 8.

$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

As $V_4 = \{e, a, b, c\}$ $a^2 = b^2 = c^2 = e$, $ab = ba = c$

$ac = ca = b$, $bc = cb = a$

7. A group of order n is isomorphic to
   (a) a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$. (b) a subgroup of A_n.
   (c) a subgroup of D_n. (d) a subgroup of $\mathbb{Z}_{2n}$

Every group is isomorphic with a permutation group of same order

8. $\mathbb{Z}_3$ is isomorphic to the following subgroup of S_3
   (a) $< (12) >$. (b) $< (13) >$ (c) A_3 (d) S_3 itself.

$\langle (12) \rangle = \langle I, (12) \rangle$ | (12) , $(12)^2 = (12)(12) = I$ | $O(A_3) = \frac{6}{2} = 3$ $\therefore A_3 \cong \mathbb{Z}_3$

$\langle (12) \rangle$ | (12), $(12) = (12)(12) = I$ | $O(A_3) = \frac{6}{2} = 3$
$= \langle I, (12) \rangle$ | $\therefore A_3 \cong Z_3$

9. A group of order 4 in which every element satisfies the equation $x^2 = e$ is isomorphic to
(a) $\mathbb{Z}_2 \times \mathbb{Z}_2$. (b) μ_4, the group of forth roots of unity under multiplication.
(c) $(\mathbb{Z}_4, +)$ (d) $\{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$.

Solution :- Option -(a)

A group of order 4 with every element = 4 is Klein's 4 group and we know $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (0,1), (1,0), (1,1)\}$, with component wise addition is also Klein's 4 group. So it will isomorphic with option (a).

10. The smallest positive integer n for which there are two non-isomorphic groups of order n equals.
(a) 2 (b) 4 (c) 6 (d) 8

Solution :- Option (b)

We know by theorem that there are two non isomorphic groups of order 4 and 6 . Hence the smallest positive integer n for which there are two non isomorphic groups of order n equals = 4

11. For each positive integer n,
(a) There is a cyclic group of order n. (b) There are two non-isomorphic groups of order n.
(c) There is a non-abelian group of order n. (d) The number of non-isomorphic groups of order n is equal to n

Solution :- Option (a)

Theorem says that for each positive integer n there is a cyclic group of order n .

12. A non-cyclic group of order 6 is isomorphic to
(a) $\mathbb{Z}_3 \times \mathbb{Z}_2$ (b) μ_6, the group of sixth roots of unity under multiplication.
(c) $U(14) = \{\bar{1}, \bar{3}, \bar{5}, \bar{9}, \bar{11}, \bar{13}\}$. (d) S_3

Solution :- Option (d)

Out of all group only the non cyclic group of order 6 is = S_3 .
Remaining all there groups are cyclic group of order 6.

13. Let $G_1 = \mathbb{Z}_3 \times \mathbb{Z}_5, G_2 = \mathbb{Z}_3 \times \mathbb{Z}_9$. Then
(a) G_1 is isomorphic to $\mathbb{Z}_{15}$ and G_2 is isomorphic to $\mathbb{Z}_{27}$.
(b) G_1 and G_2 are not isomorphic to $\mathbb{Z}_{15}$, $\mathbb{Z}_{27}$ respectively.
(c) G_1 is not isomorphic to $\mathbb{Z}_{15}$ but G_2 is isomorphic to $\mathbb{Z}_{27}$
(d) G_1 is isomorphic to $\mathbb{Z}_{15}$ but G_2 is not isomorphic to $\mathbb{Z}_{27}$

Solution :- Option (d)

By then properties of external direct product we know that
$\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ when ever (m , n) =1 i.e. m and n are relatively

prime .
We know , (3 , 5) = 1 then $\mathbb{Z}_3 \times \mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_{15}$
But , (3 , 9) = 3 , then $\mathbb{Z}_3 \times \mathbb{Z}_9$ is not isomorphic to $\mathbb{Z}_{27}$.

14. The number of elements of order 4 in $\mathbb{Z}_8 \times \mathbb{Z}_4$ is
(a) 4 (b) 8 (c) 20 (d) 16

Solution :- Option (b)
The group $\mathbb{Z}_8 = \{ 0 , 1 , 2 , 3 , 4 , 5 , 6 , 7 \} \; mod \; 8$
$\mathbb{Z}_4 = \{ 0 , 1 , 2 , 3 \} \; mod \; 4$
The order of the element (a , b) = 4 = lcm(O(a) , O(b)) where a ∈ $\mathbb{Z}_8$ and b ∈ $\mathbb{Z}_4$.
Now the possibilities are :-
(i) O(a) = 1 and O(b) = 4
(ii) O(a) = 4 and O(b) = 1
(iii) O(a) = 4 = O(b)
As l.c.m (a , b) = { 1 , 4} = { 4, 1} = { 4 ,4 } = 4
By the theorem we know that the number of elements of order n in the group $\mathbb{Z}_m$ is $\phi(n)$.
So number of order 4 elements in $\mathbb{Z}_8$ and $\mathbb{Z}_4$ is $\phi(4) = 2$
Therefore the order of 0 in $\mathbb{Z}_8$ is 1 and order of 2 and 6 in $\mathbb{Z}_8$ is 4
Similarly , the order of 0 in $\mathbb{Z}_4$ is 1 and order of 1 and 3 in $\mathbb{Z}_4$ is 4
Therefore , order 4 elements in $\mathbb{Z}_8 \times \mathbb{Z}_4$ is
(0 ,1) , (0 , 3) , (2 , 0) , (6 , 0) , (2 , 1) , (2 , 3) , (6 , 1) , (6 , 3)

15. Consider the following groups i) $\mathbb{Z}_4$ ii) $U(10)$ ii) $U(8)$ iv) $U(5)$. The only non-isomorphic group among them is
(a) $U(8)$ (b) $U(10)$ (c) $\mathbb{Z}_4$ (d) All are isomorphic.

Solution :-
Option (a)
$\mathbb{Z}_4$ is a cyclic group of order 4 .
U(10) is also a cyclic group of order 4.
U(5) is also a cyclic group of order 4.
But , U(8) is a noncyclic Klein's 4 group of order 4 .

16. Consider the following groups i) S_3 ii) μ_6 ii) $\mathbb{Z}_6$ iv) $\mathbb{Z}_2 \times \mathbb{Z}_3$ v) $U(9)$. The only non-isomorphic group among them is
(a) S_3 (b) μ_6 (c) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (d) $S_3 \simeq U(9)$ and $\mu_6, \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_3$ are isomorphic. .

Solution :- Option (a)
Here , $\mu_6 = 6^{th}$ root of unity , is a cyclic group of order 6.
$\mathbb{Z}_2 \times \mathbb{Z}_3$, is also a cyclic group of order $2 \times 3 = 6$.
$\mathbb{Z}_6$, is also a cyclic group of order 6 .

U(9) = { 1 , 2 ,4 , 5 , 7 , 8 } is also cyclic group of order 6 with one of the generator = 2.
But as we know S_3 =symmetric group of 3 symbol is a non cyclic group of order 6 .

17. If for positive integers m, n have $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to$(\mathbb{Z}_{mn}, +)$ then which is not true,

(a) m, n are relatively prime.

(b) m, n are odd.

(c) m, n are prime.

(d) $m = p^r, n = q^s$ for primes p, q and $r, s \in \mathbb{N}$.

Solution :-
Option (a)
By then properties of external direct product we know that
$\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$ when ever (m , n) =1 i.e. m and n are relatively prime .

18. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H = \mathbb{Z}_4 \times \{\bar{0}, \bar{1}\}, K =< (\bar{1}, \bar{2}) >$ be subgroups of G Then
(a) G/H is isomorphic to G/K (b) G/H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$
(c) H and K are isomorphic. (d) none of these.

Solution :-
Option (d)
Here G = $\mathbb{Z}_4 \times \mathbb{Z}_4$, where O(G) = $4 \times 4 = 16$
Now , H = $\mathbb{Z}_4 \times \{ \bar{0}, \bar{1} \} = \{ \bar{0}, \bar{1}, \bar{2}, \bar{3} \} \times \{ \bar{0}, \bar{1} \}$
$= \{ (\bar{0},\bar{0}), (\bar{0},\bar{1}), (\bar{1},\bar{0}), (\bar{1},\bar{1}), (\bar{2},\bar{0}), (\bar{2},\bar{1}), (\bar{3},\bar{0}), (\bar{3},\bar{1}) \}$
Hence , O(H) = 8
Therefore , O(G) / O(H) = 16/8 = 2
Now , let us consider ,
K = < $(\bar{1},\bar{2}) > = < (\bar{1},\bar{2}), (\bar{2},\bar{0}), (\bar{3},\bar{2}), (\bar{0},\bar{0}) \}$, Hence = O(K) = 4
Therefore , O(G) / O(H) = 16/4 = 4
Hence , G / H ≇ G / K ,
Now , O ($\mathbb{Z}_2 \times \mathbb{Z}_2$) = 4
But , $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{ (0,0), (0,1), (1,0), (1,1)\}$, with component wise addition is also Klein's 4 group .
Hence , $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to G / H
Order of H and K is not same so they can not be isomorphic.

19. From the given list of pairs of group, pick the pair of non-isomorphic groups
(a) $3\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}_4$ (b) $8\mathbb{Z}/48\mathbb{Z}$ and $\mathbb{Z}_6$
(c) $\mathbb{Z}_4$ and V_4 (d) $(\mathbb{Z} \times \mathbb{Z})/(2\mathbb{Z} \times 2\mathbb{Z})$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

Solution :-
Option (c)
Here , $3\mathbb{Z} / 12\mathbb{Z}$ and $\mathbb{Z}_4$ both are cyclic group of order 4 . Hence they are isomorphic to each other .
$8\mathbb{Z} / 48\mathbb{Z}$ and $\mathbb{Z}_6$ both are cyclic group of order 6 . Hence they are isomorphic to each other .
$(\mathbb{Z} \times \mathbb{Z}) / (2\mathbb{Z} \times 2\mathbb{Z})$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ both are cyclic group of order 4.
Hence they are isomorphic to each other .
But , $\mathbb{Z}_2$ and V_4 both are group of order 4 but $\mathbb{Z}_2$ is cyclic where as V_4 is non cyclic group of order 4.
Hence they are not isomorphic to each other .

20. From the given list of pairs of groups, pick the pairs of isomorphic groups
(a) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ (b) $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$
(c) D_4 and $\mathbb{Z}_4 \times \mathbb{Z}_2$ (d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $V_4 \times \mathbb{Z}_2$

Solution :- option (d)
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is a group non cyclic of order $2 \times 2 \times 2 = 8$ as the group is having elements only of order 2.
And , $\mathbb{Z}_4 \times \mathbb{Z}_2$ is a group of order $4 \times 2 = 8$

, $\mathbb{Z}_4 \times \mathbb{Z}_2$ has an element of order 4 but is not cyclic,

The set , $\mathbb{Z}_4 \times \mathbb{Z}_2$ is given as

$$\{(0,0),(1,0),(2,0),(3,0),(0,1),(1,1),(2,1),(3,1)\}$$

$\mathbb{Z}_4 \times \mathbb{Z}_2$ has 4 elements of order 4 and 2 elements of order 2 and 1 element of order 1.

$\mathbb{Z}_8$ is cyclic of order 8,

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \ncong \mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2 \ncong \mathbb{Z}_8$

The group D_4 of symmetries of the square is a nonabelian group of order 8.

D_4 has 4 elements of order 2 and 3 elements of order 4 and 1 element of order 1.
Similarly , $D_4 \ncong \mathbb{Z}_8$
But , $V_4 \times \mathbb{Z}_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

21. If G, H, K are finite Abelian groups and $G \times K \cong H \times K$, then

(a) $G = H$

(b) G need not be isomorphic to H

(c) $G \cong H$

(d) None of these.

Solution :- Option (c)
Here , G , H , K are finite abelian groups and $G \times K \cong H \times K$ then $G \cong H$.

22. If G is an Abelian group with order mn where $(m, n) = 1$. $G(m) = \{g \in G : g^m = e\}$ and $G(n) = \{g \in G : g^n = e\}$, then
   (a) $G = G(n) \cup G(m)$
   (b) $G \cong G(n) \times G(m)$
   (c) $G = G(m)G(m)$
   (d) None of these.

Solution :- Option (b)
Here , (m ,n) = 1 with that , G(m) = { $g \in G \, : g^m = e$ } and
G(n) = { g $\in G : g^n = e\}$
If G be an abelian group of order mn then obviously
G $\cong G(m) \times G(n)$

24. $U(16)/ < \bar{9} >$ is isomorphic to
(a) $\mathbb{Z}_4$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\mathbb{Z}_8$ (d) Non of these.

Solution :-
Option (a)
Here , U(16) = { 1 , 3, 5, 7,9, 11, 13, 15}
< $\bar{9} > = \{ \bar{9}, \bar{1} \}$
U(16) / < $\bar{9} > \; = \; 8/2 = 4$
As U(16) and < $\bar{9} >$ both are cyclic then U(16) / $< \bar{9} >$ is also cyclic group of order 4 . Then U(16) / $< \bar{9} > \cong \mathbb{Z}_4$

25

If $\mathbb{R}^*$ and $\mathbb{R}^+$ are multiplicative groups and $f : \mathbb{R}^* \to \mathbb{R}^+$ is defined by $f(x) = |x|$ then
   (a) f is an injective homomorphism.
   (b) f is not a homomorphism.
   (c) $\ker f = \{-1, 1\}$.
   (d) f is an isomorphism.

Solution :-
Option (c)
Let us find find the kernal of $f : \mathbb{R}^* \to \mathbb{R}^+$ as f(x) = | x |
So , ker f = { $x \in \mathbb{R}^*$ *such that* $f(x) = 1$ }
= { $x \in \mathbb{R}^*$ *such that* $|x| = 1\}$
= { -1 , 1 }