OPERATION RESEARCH

LINEAR PROGRAMMING PROBLEM

Linear Programming is a technique for determining an optimum schedule of independent activities in view of the available resources. Programming is just another word for 'planning' and refers to the process of determining a particular plan of action from amongst several alternatives. The word 'linear' stands for indicating that all relationships involved in a particular problem are linear.

LPP consists of three components, namely , (i) the decision variables (activities) , (ii) the objective (goal) and (iii) the constraints (restrictions).

(i) The decision variables refer to the activities that are competing one another for sharing the resources available. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. All the decision variables are considered as continuous, controllable and non-negative.

(ii) A linear programming problem must have an objective which should be clearly identifiable and measurable in quantitative terms. It could be of profit (sales) maximisation, cost (time) minimisation and so on. The relationship among the variables representing objective must be linear.

(iii) There are always certain limitations (or constraints) on the use of resources such as labour, space, raw materials, money, etc. that limit the degree to which an objective can be achieved. Such constraints must be expressed as linear inequalities or equalities in terms of decision variables.

BASIC ASSUMPTIONS

The following four basic assumptions are necessary for all linear programming problem:

(i) **Certainty**- In all LPP's it is assumed that all the parameters such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit decision variable must be known and fixed. It means that all the coefficients in the objective function as well as in the constraints are completely known with certainty and do not change during the period of study.

(ii) **Divisibility** (or continuity)- This implies that solution values of the decision variables and resources can take on any non-negative values including fractional values of the decision variables. For example: It is possible to produce 4.3 quintals of wheat or 17.35 thousand kilometres of cloth or 6.52 kilolitres of milk. Therefore, these variables are divisible but it is not possible to produce 2.6 refrigerators. Such variables are not divisible and hence are to be assigned integer values. When it is necessary to have integer variables, the integer programming problem is considered to attain the desired values.

(iii) **Proportionality**- This requires the contribution of each decision variable in both the objective function and the constraints to be directly proportional to the value of the variable. For example: if the production of 1 unit of a particular product uses 3hours of a particular resource then the production of 6 units of that product uses 3x6, i.e., 18hours of this resource.

(iv) **Additivity**- the value of the objective function for the given values of decision variables and the total sum of resources used must be equal to the sum of the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision variable respectively. For example: the total profit earned by the sale of 2 products A and B must be equal to the sum of the profits earned separately by A and B similarly, the amount of a resource consumed by A and B must be equal to the sum of resources used for A and B individually.

MATHEMATICAL FORMULATION OF A PROBLEM

The procedure for mathematical formulation of a linear programming problem consists of the following major steps:

STEP1- Study the given situation to find the key decisions to be made.

STEP2- Identify the variables involved and designate them by symbols. x_j (j = 1, 2, 3, …)

STEP3- State the feasible alternatives which generally are: $x_j \geq 0$, for all j.

STEP4- Identify the constraints in the problem and express them as linear inequalities or equation, LHS on which are linear functions of the decision variables.

STEP5- Identify the objective function and express it as a linear function of the decision variables.

For example:

A company has three operational departments (weaving, processing and packing) with capacity to produce three different types of clothes namely suitings, shirtings and woollens yielding a profit of Rs.2, Rs.4 and Rs.3 per metre respectively. One metre of suiting requires 3 minutes in weaving, 2 minutes in processing and 1 minute in packing. Similarly one metre of shirting requires 4 minutes in weaving, 1 minute in processing and 3 minutes in packing. One metre of woollen requires 3 minutes in each department. In a week , total run time of each department is 60, 40 and 80 hours for weaving, processing and packing respectively.

Formulae the linear programming problem to find the product mix to maximize the profit.

Mathematical Formulation - Summary of the data

	Departments			*Profit (Rs. per metre)*
	Weaving (in minutes)	*Processing (in minutes)*	*Packing (in minutes)*	
Suitings	3	2	1	2
Shirtings	4	1	3	4
Woollens	3	3	3	3
Availability (minutes)	60 x 60	40 x 60	80 x 60	

Mathematical Format :

Find x_1, x_2 and x_3 so as to maximize

$$Z = 2x_1 + 4x_2 + 3x_3$$

Subject to the constraints:

$$3x_1 + 4x_2 + 3x_3 \leq 3600$$
$$2x_1 + x_2 + 3x_3 \leq 2400$$
$$x_1 + 3x_2 + 3x_3 \leq 4800$$
$$x_1 \geq 0,\quad x_2 \geq 0 \text{ and } x_3 \geq 0$$

An electronic company is engaged in the production of two components C_1 and C_2 used in T.V. sets. Each unit of C_1 costs the company Rs. 25 in wages and Rs. 25 in material , while each unit of C_2 costs the company Rs. 125 in wages and Rs.75 in material. The company sells both products on one-period credit terms, but the company's labour and material expenses must be paid in cash. The selling price of C_1 is Rs.150 per unit and of C_2 is Rs.350 per unit. Because of the strong monopoly of the company for these components, it is assumed that the company can sell at the prevailing prices as many units as it produces. The company's production capacity is, however, limited by two considerations. First, at the beginning of period 1, the company has an initial balance of Rs.20,000(cash plus bank credit plus collections from past credit sales). Second, the company has available in each period 4,000 hours of machine time and 2,800 hours of assembly time. The production of each C_1 requires 6 hours of machine time and 4 hours of assembly time, whereas the production of each C_2 requires 4 hours of machine time and 6 hours of assembly time. Formulate this problem as an Linear Programming model so as to maximize the total profit to the company.

Problem 3

The manager of an oil refinery must decide on the optimum mix of two possible blending processes of which the input and output production runs are as follows:

Process	Input		Output	
	Crude A	Crude B	Gasoline X	Gasoline Y
1	6	4	6	9
2	5	6	5	5

The maximum amounts available of crudes A and B are 250 units and 200 units respectively. Market demand shows that at least 150 units of gasoline X and 130 units of gasoline Y must be produced. The profits per production run from process 1 and process 2 are Rs. 4 and Rs. 5 respectively. Formulate the problem for maximising the profit.

Solution:

Mathematical formulation:

Step 1: The key decision is to determine the number of units of gasoline produced from process 1 and process 2.

Step 2: Decision variables : Let X_1, X_2 = number of units of gasoline produced from process 1 and process 2 respectively.

Step 3: Feasible alternatives : $X_1 \geq 0$, $X_2 \geq 0$.

Step 4: The constraints are on the availability of crude oil and demand of crude oil, viz.,

$6X_1 + 5X_2 \leq 250$ (Availability of crude A)

$4X_1 + 6X_2 \leq 200$ (Availability of crude B)

$6X_1 + 5X_2 \geq 150$ (Demand of gasoline X)

$9X_1 + 5X_2 \geq 130$ (Demand of gasoline Y)

Step 5: The objective is to maximize the total profit from the production of gasoline, viz.,

$4X_1 + 5X_2$

Hence, the required linear programming problem is

Maximize $z = 4X_1 + 5X_2$

Subject to the constraints :

$6X_1 + 5X_2 \leq 250$

$4X_1 + 6X_2 \leq 200$

$6X_1 + 5X_2 \geq 150$

$9X_1 + 5X_2 \geq 130$

$X_1 \geq 0$, $X_2 \geq 0$.

Problem 4

Three grades of coal A,B and C contain ash and phosphorus as impurities. In a particular industrial process a fuel obtained by blending the above grades containing not more than 25% ash and 0.03% phosphorus is required. The maximum demand of the fuel is 100 tons. Percentage impurities and costs of the various grades of coal are shown below. Assuming there is an unlimited supply of each grade of coal and there is no loss in blending, formulate the blending problem to minimise the cost.

Coal grade	% ash	% phosphorus	Cost per ton (in Rs.)
A	30	0.02	240
B	20	0.04	300
C1	35	0.03	280

MATHEMATICAL FORMULATION

Decision Variables: Let x_1 = tons of grade *A* coal,

X_2 = tons of grade *B* coal and

X_3 = tons of grade *C* coal

Objective function: Minimize $z = 240x1 + 300x2 + 280x3$

Constraints : $0.3x1 + 0.2x2 + 0.35x3 \leq o.25(x1 + x2 + x3)$

$x1 - x2 + 2x3 \leq 0$ (ash)

$0.02/100x1 + 0.04/100x2 + 0.03/100x3 \leq 0.03/100(x1+x2+x3)$

$-x1 + x2 \leq 0$ (phosphorus)

$x1 + x2 + x3 \leq 0$ (demand of fuel)

$x1 \geq 0$, $x2 \geq 0$ and $x3 \geq 0$

Problem 5

An engineering company is planning to diversify its operations during the year 2006-2007. The company has allocated capital expenditure budget to Rs. 5.15 crore in the year 2006 and Rs. 6.50 crore in the year 2007. The company has five investment expenditures of each project in the two years are as follows :

Project	Estimated net returns (in'000 Rs.)	Cash expenditure(in '000 Rs.)	
		Year 2006	Year 2007
A	240	120	320
B	390	550	594
C	80	118	202
D	150	250	340
E	182	324	474

Assume that the returns from a particular project would be in direct proportion to the investment in it, so that , for example , if in a project , say A, 20% (of 120 in 2006 and 320 in 2007) is invested, then the resulting net returns in it would be 20% (of 240). This assumption also implies that individuality of the project should be ignored. Formulate this capital budgeting problem as a Linear Programming model to maximize the net returns.

SOLUTION:

Mathematical Formulation:

Decision variables : Let x1, x2, x3, x4 and x5 represent the proportion of investment in project A, B, C, D and E respectively.

Objective function: The objective is to maximize the net returns . Thus, the objective function is:

$$\text{Maximize} \quad z = 249x1 + 390x2 + 80x3 + 150x4 + 182x5$$

Constraints : Capital expenditure budget constraints are :

$$120x1 + 550x2 + 118x3 + 250x4 + 324x5 \le 515 \qquad (\text{ for } 2006)$$

$$320x1 + 594x2 + 202x3 + 340x4 + 474x5 \le 650 \qquad (\text{ for } 2007)$$

Also, $x1 \ge 0$, $x2 \ge 0$, $x3 \ge 0$, $x4 \ge 0$ and $x5 \ge 0$ (Non-negative restrictions)

Problem 6

A city hospital has the following minimal daily requirements for nurses:

Period	Clock time (24 hr day)	Minimal number of nurses required
1	6 am - 10 am	2
2	10 am - 2 pm	7
3	2 pm - 6 pm	15
4	6 pm - 10 pm	8
5	10 pm - 2 am	20
6	2 am - 6 am	6

Nurses report to the hospital at the beginning of each period and work for 8 consecutive hours. The hospital wants to determine the minimal number of nurses to be employed so that there will be sufficient number of nurses available for each period. Formulate this problem as a linear programming model by setting up appropriate constraints and objective function.

Mathematical Formulation:

The key decision to be made is to determine the number of nurses to be employed in the city hospital.

Decision variables: Let x_j (j = 1, 2, 3, 4, 5, 6) be the number of nurses reporting to work on shift j.

Also, since negative values of x_j are meaningless, we must have $x_j \geq 0$ for all j.

Objective function: If the values of nurses reporting to work on shifts 1, 2, 3…..6 is known, then the manpower schedule can be decided. Thus, the objective is to minimize the total number of nurses (manpower for all six periods together). As such, the objective is to minimize the linear function :

$$Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Constraints: In any 8-hour period , the number of nurses reporting at the beginning of the period and the number of nurses continuing from the earlier period should be at least equal to the minimum requirement.

Thus, we have the following constraints:

$X_1 + x_2 \geq 7$, $x_2 + x_3 \geq 15$, $x_3 + x_4 \geq 8$

$X_4 + x_5 \geq 20$, $x_5 + x_6 \geq 6$, $x_6 + x_1 \geq 2$

Thus, the given LPP is:

Minimize $z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Subject to the constraints:

$X_1 + x_2 \geq 7$, $x_2 + x_3 \geq 15$, $x_3 + x_4 \geq 8$

$X_4 + x_5 \geq 20$, $x_5 + x_6 \geq 6$, $x_6 + x_1 \geq 2$

And $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$, $x_5 \geq 0$ & $x_6 \geq 0$