1

MATRICES

UNIT STRUCTURE

1.0 OBJECTIVES

In this chapter a student has to learn the

- Concept of Adjoint of a matrix.
- Inverse of a matrix.
- Rank of a matrix and methods finding these.

1.1 INTRODUCTION

At higher secondary level, we have studied the definition of a matrix, operations on the matrices, types of matrices inverse of a matrix etc.

In this chapter, we are studying Adjoint method of finding the inverse of a square metrix and also the rank of a matrix.

1.2 DEFINITIONS

a) Minor of an element :

Consider a square matrix A of order n
Let

$$A = \left[a_{ij}\right]_{n\times n}$$

The matrix is also can be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & - & - & - & a_{1n} \\ a_{21} & a_{22} & a_{23} & - & - & - & a_{2n} \\ - & - & - & - & - & - & - \\ - & - & - & - & - & - & - \\ a_{n1} & a_{n2} & a_{n3} & - & - & - & a_{nn} \end{bmatrix}$$

Minor of an element a_{ij} is a determinant of order (nd) by deleting the elements of the metrix A, which are in ith row and jth column of A.

E.g. Consider,

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

M_{11} = Minor of an element a_{11}

$$A = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

II y

$$M_{12} = \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{bmatrix}$$

E.g. (ii) Let,

$$A = \begin{bmatrix} 2 & 5 & 8 \\ 1 & 3 & 2 \\ 0 & 4 & 6 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} 3 & 2 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 1 & 2 \\ 0 & 6 \end{bmatrix}, M_{13} = \begin{bmatrix} 1 & 3 \\ 0 & 4 \end{bmatrix}$$

$$M_{21} = \begin{bmatrix} 5 & 8 \\ 4 & 6 \end{bmatrix}, M_{12} = \begin{bmatrix} 2 & 8 \\ 0 & 6 \end{bmatrix}, M_{23} = \begin{bmatrix} 2 & 5 \\ 0 & 4 \end{bmatrix}$$

(b) Cofactor of an element :-

If $A = \left[a_{ij}\right]$ is a square matrix of order n and a_{ij} denotes confactor of the element a_{ij}.

$C_{ij} = (-1)^{i+j} \, . \, M_{ij}$, Where M_{ij} is minor of a_{ij}.

E.g. Consider,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix}$$

$$c_{11} = (-1)^{1+1} M_{11} \qquad\qquad c_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 3 & 6 \end{vmatrix}$$

$$= (-1)^{1+1} . \begin{vmatrix} 2 & 1 \\ 7 & 6 \end{vmatrix} \qquad\qquad = (-1)^{3} \times (0-3)$$

$$= (1) \times (12-7) \qquad\qquad = (-1) \times (-3)$$

$$= 5 \qquad\qquad = 3$$

(C) Cofactor Matrix :-

A matix $C = [C_{ij}]$ where C_{ij} denotes cofactor of the element a_{ij}. Of a matrix A of order nxn, is called a cofactor matrix.

In above matrix A, cofactor matrix is

$$C = \begin{bmatrix} 5 & 3 & -6 \\ 10 & -6 & 9 \\ -3 & -1 & 2 \end{bmatrix}$$

Similarly for a matrix, $A = \begin{bmatrix} 1 & 2 \\ 3 & 9 \end{bmatrix}$ the cofactor matrix is $c= \begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$

(d) Adjoint of Matrix :-

If A is any square matrix then transpose of its cofactor matrix is called Adjoint of A.

Thus in the notations used,

$$\text{Adjoint of } A = C^1$$

Adjoint of a matrix A is denoted as Adj.A

Thus if,

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 2 & 1 \\ 3 & 7 & 6 \end{bmatrix} \text{ then Adj. A} = \begin{bmatrix} 5 & -10 & 3 \\ 3 & -6 & -1 \\ -6 & 9 & 2 \end{bmatrix}$$

(d) Inverse of a square Matrix:-

Two non-singular square matrices of order n A and B are said to be inverse of each other if,

AB=BA=I, where I is an identity matrix of order n.

Inverse of A is denoted as A^{-1} and read as A inverse.

Thus

$$AA^{-1}=A^{-1}A=I$$

Inverse of a matrix can also be calculated by the Formula.

$A^{-1} = \frac{1}{|A|}$ Adj.A where $|A|$ denotes determinant of A.

1.3 ILLUSTRATIVE EXAMPLES

Ex) Find the inverse of matrix A by Adjoint method, if

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

Sol) Consider

$$|A| = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$= 0(-1)-1(-8)+2(-5)$$

$$= 0+8-10$$

$$= -2$$

Co factor of the elements of A are as follows

$$C_{11} = (-1)^{1+1}.\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = -1$$

$$C_{12} = (-1)^{1+2}.\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = 8$$

$$C_{13} = (-1)^{1+3}.\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$C_{21} = (-1)^{2+1}.\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$C_{22} = (-1)^{2+2}.\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} = -6$$

$$C_{23} = (-1)^{2+3}.\begin{vmatrix} 0 & 1 \\ 3 & 1 \end{vmatrix} = 3$$

$$C_{31} = (-1)^{3+1}.\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1$$

$$C_{32} = (-1)^{3+2}.\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = 2$$

$$C_{33} = (-1)^{3+3}.\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$$

Thus,

Cofactor of matrix $C = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & 1 \end{bmatrix}$

And Adjoint of A= C^1

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & 1 \end{bmatrix}$$

Ex 2) Find the given matrix, $|A| = 27$

Cofactor matrix $C = \begin{bmatrix} 6 & -3 & 3 \\ 6 & 15 & -6 \\ -3 & 3 & 6 \end{bmatrix}$

$$\therefore \text{ Adj. } A = \begin{bmatrix} 6 & 6 & 3 \\ -3 & 15 & 3 \\ 3 & -6 & 6 \end{bmatrix}$$

Now, A^{-1} $A^{-1} = \frac{1}{|A|} \cdot \text{Adj.A}$

$$= \frac{1}{27}\begin{bmatrix} 6 & 6 & 3 \\ -3 & 15 & 3 \\ 3 & -6 & 6 \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \\ -1 & 5 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Exercise 1.1

Q. 1) Find the inverse of the following matrices using Adjoint method, if they exist.

i) $\begin{vmatrix} 1 & 2 \\ 2 & -2 \end{vmatrix}$, ii) $\begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix}$, iii) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$,

iv) $\begin{vmatrix} 1 & 3 & -2 \\ -3 & 0 & -5 \\ 2 & 5 & 0 \end{vmatrix}$, v) $\begin{vmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$, vi) $\begin{vmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{vmatrix}$

vii) $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{vmatrix}$

Q.3) If A = $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$, B = $\begin{vmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{vmatrix}$, C= $\begin{vmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{vmatrix}$, ,

prove that A= B.C^{-1}

Q. 4) If A = $\begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & 3 \end{bmatrix}$, prove that Adj. A= A

Q. 5) If A = $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, verify if $(Adj.A)^1$= $(Adj.A^1)$

Q.6) Find the inverse of $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 2 & 3 \end{bmatrix}$, hence find inverse of

$$A = \begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 6 & 6 & 9 \end{bmatrix}$$

1.4 RANK OF A MATRIX

a) Minor of a matrix

Let A be any given matrix of order mxn. The determinant of any submatrix of a square order is called minor of the matrix A.

We observe that, if 'r' denotes the order of a minor of a matrix of order mxn then $1 \le r \le m$ if m<n and $1 \le r \le n$ if n<m.

e.g. Let

$$A = \begin{bmatrix} 1 & 3 & -1 & 4 \\ 4 & 0 & 1 & 7 \\ 8 & 5 & 4 & -3 \end{bmatrix}$$

The determinants $\begin{bmatrix} 1 & 3 & -1 \\ 4 & 0 & 1 \\ 8 & 5 & 4 \end{bmatrix}$, $\begin{bmatrix} 3 & -1 & 4 \\ 0 & 1 & 7 \\ 5 & 4 & -3 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 4 \\ 4 & 1 & 7 \\ 8 & 4 & -3 \end{bmatrix}$,

$\begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix}$, $\begin{vmatrix} 0 & 1 \\ 5 & 4 \end{vmatrix}$, $\begin{vmatrix} 3 & 4 \\ 0 & 7 \end{vmatrix}$, $|1|$, $|0|$, $|-3|$,

Are some examples of minors of A.

b) Definition – Rank of a matrix

A number 'r' is called rank of a matrix of order mxn if there is almost one minor of the matrix which is of order r whose value is non-zero and all the minors of order greater than 'r' will be zero.

e.g.(i) Let

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix}$$

Consider e.g. Let $A_1 = \begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4,\ A_2 = \begin{vmatrix} 0 & 2 \\ 4 & 1 \end{vmatrix} = -8$ etc.

$$A_3 = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 4 & 1 \\ 3 & 5 & 7 \end{bmatrix} = 1(23) + 2(-2) = 19 \neq 0$$

$\therefore$ Rank of A= 3

(ii) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Here,

$$A_1 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = 1(1) - 1(-1) + 2(-1) = 0$$

$$A_2 = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$$

Thus minor of order 3 is zero and atleast one minor of order 2 is non-zero
$\therefore$ Rank of A = 2.

Some results:

(i) Rank of null matrix is always zero.

(ii) Rank of any non-zero matrix is always greater than or equal to 1.

(iii) If A is any mxn non-zero matrix then Rank of A is always equal to rank of A.

(iv) Rank of transpose of matrix A is always equal to rank of A.

(v) Rank of product of two matrices cannot exceed the rank of both of the matrices.

(vi) Rank of a matrix remains unleasted by **elementary transformations**.

Elementary Transformations:

Following changes made in the elements of any matrix are called elementary transactions.

(i) Interchanging any two rows (or columns) .

(ii) Multiplying all the elements of any row (or column) by a non-zero real number.

(iii) Adding non-zero scalar multitudes of all the elements of any row (or columns) into the corresponding elements of any another row (or column).

1.5 CANONICAL FORM OR NORMAL FORM

If a matrix A of order mxn is reduced to the form $\begin{bmatrix} I_r & o \\ o & o \end{bmatrix}$ using a sequence of elementary transformations then it called canonical or normal form. Ir denotes identity metrix of order ‘r’ .

Note:- If any given matrix of order mxn can be reduced to the canonical form which includes an identity metrix of order ‘r’ then the matrix is of rank ‘r’.

e.g. (1) Consider

$$A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & 6 \end{bmatrix}$$

$$R_2 - 3R_1,\ R_3 - 2R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$$R_2 - 7R_3$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 33 & 66 \\ 0 & -1 & -5 & -10 \end{bmatrix}$$

$$R_1 - R_2,\ R_3 + R_2$$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 28 & -56 \end{bmatrix}$$

$$R_3 \times \frac{1}{28}$$

$$\sim \begin{bmatrix} 1 & 0 & -32 & -64 \\ 0 & 1 & 33 & 66 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$R_1 + 32\,R_3,\ \ R_2 - 33\,R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & o \end{bmatrix}$$

$\therefore$ Rank of A=3

e.g. (2) Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{bmatrix}$$

$$R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 - 3R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_2 - 2C_1$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C_1 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \end{bmatrix}$$

$\therefore$ Rank of A= 2

(3)

$$A = \begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & -1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$$R_2 - 2R_1,\ R_3 - 3R_1,\ R_4 - 6R_1,$$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_2 - R_3$

$$\sim \begin{bmatrix} 1 & -1 & -2 & -4 \\ 0 & 1 & -6 & -3 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{bmatrix}$$

$R_1 + R_2,\ R_3 - 4R_2,\ R_4 - 9R_2$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 66 & 44 \end{bmatrix}$$

$R_4 - 2R_3$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -3 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \times \frac{1}{11}$

$$\sim \begin{bmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & -6 & -2 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 - C_4$

$$\sim \begin{bmatrix} 1 & 0 & -1 & -7 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 + R_3,\ R_2 + 3R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 - (5C_1 + 3C_2 + 2C_2)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of A= 3

Exercise:-

Reduce the following to normal form and hence find the ranks of the matrices.

i) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$, ii) $\begin{vmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{vmatrix}$ iii) $\begin{vmatrix} -3 & 4 & 6 \\ 5 & -5 & 7 \\ 3 & 1 & -4 \end{vmatrix}$

iv) $\begin{vmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{vmatrix}$ v) $\begin{vmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{vmatrix}$ vi) $\begin{vmatrix} 1 & 2 & 1 & 0 \\ 3 & 2 & 1 & 2 \\ 2 & -1 & 2 & 5 \\ 5 & 6 & 3 & 2 \\ 1 & 3 & -1 & -3 \end{vmatrix}$

vii) $\begin{vmatrix} 2 & 6 & -2 & 6 & 10 \\ -3 & 3 & -3 & -3 & -3 \\ 1 & -2 & 4 & 3 & 5 \\ 2 & 0 & 4 & 6 & 10 \\ 1 & 0 & 2 & 3 & 5 \end{vmatrix}$ viii) $\begin{vmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{vmatrix}$

1.6 NORMAL FORM PAQ

If A is any mxn matrix 'r' then there exist non singular matrices P and Q such that,

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

We observe that, the matrix A can be expressed as

A = Im In …………(i)

Where Im In are the identity matrices of order m and n respectively. Applying the elementary transformations on this equation. A in L.H.S. can be reduced to normal form. The equation can be transformal into the equations.

$$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = PAQ \text{ …………(ii)}$$

Note that, the row operations can be performed simultaneously on L.H.S. and prefactor (i.e. Im in equation (i)) and column operations can be performed simultaneously on L.H.S. and post factor in R.H.S. i.e. [(In in eqn (i)]

Examples :-

Find the non-singular matrices P and Q such that PAQ is in normal and hence find the rank of A.

(i) $$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix}$$

Solutions : Consider

$$A = I_3 A I_3$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 4 & -1 \\ 1 & 5 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 5 & -4 \\ 3 & 4 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_2 - 5CS_1,\ C_3 + 4C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & -11 & -11 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & -11 & -11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -5 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 - R_1,\ R_3 - 2R_1,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -11 & 11 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & 5 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$C_3 + C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -11 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \times \frac{1}{11},$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & -1 \\ 1/11 & 0 & -2/11 \end{bmatrix} A \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1/11 & 0 & 2/11 \\ -1 & 1 & -1 \end{bmatrix} A \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus

$$P = \begin{bmatrix} 0 & 0 & 1 \\ -1/11 & 0 & 2/11 \\ -1 & 1 & -1 \end{bmatrix} \quad \Delta \quad |P| = \frac{-1}{11}$$

$$Q = \begin{bmatrix} 1 & -5 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \Delta \quad |Q| = 1$$

P and Q are non-singular matrices
Also Rank of A = 2

ii) $A = \begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

Solutions:
Consider:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_2 - C_1,\ C_3 - C_1,\ C_4 - 2C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & -6 & -2 & -4 \\ 2 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 3R_1,\ R_3 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 - 6R_3,$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 56 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_4 - 2C_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$C_3 - 5C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 28 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & 1 & 9 \\ 1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \times \frac{1}{28},\ R_3 \times (-1)$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 3/14 & 1/28 & 9/28 \\ -1 & 0 & -2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[I_3 \quad 0] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 3/14 & 1/28 & 9/28 \end{bmatrix}, \ |P| = \frac{1}{28}$$

$$Q = \begin{bmatrix} 1 & -1 & 4 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, |Q| = 1$$

$\therefore$ P&Q are non singular.
Also,
Rank of A = 3.

Exercise:
Find the non-singular matrices P and Q such that PAQ is in normal form and hence find rank of matrix A.

i) $\begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & -4 \\ 3 & 3 & -6 \end{bmatrix}$ ii) $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$ iii) $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

iv) $\begin{bmatrix} 2 & 3 & 4 & 7 \\ -3 & 4 & 7 & -9 \\ 5 & 4 & 6 & -5 \end{bmatrix}$ (v) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$

5

VECTOR CALCULAS

UNIT STRUCTURE

5.0 OBJECTIVES

After going through this unit, you will be able to

- Learn vector differentiation.
- Operators, del, grad and curl.
- Properties of operators

5.1 INTRODUCTION

Vector algebra deals with addition, subtraction and multiplication of vertex. In vector calculus we shall study differentiation of vectors functions, gradient, divergence and curl.

Vector:

Vector is a physical quantity which required magnitude and direction both.

Unit Vector:

Unit Vector is a vector which has magnitude 1. Unit vectors along co-ordinate axis are $\hat{i}$ and $\hat{j}$, $\hat{k}$ respectively.

$$\left|\hat{i}\right| = \left|\hat{j}\right| = \left|\hat{k}\right| = 1$$

Scalar Triple Vector:

Scalar triple product of three vectors is defined as $\bar{a}.\left(\bar{b}\times\bar{c}\right)$ or $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$. Geometrical meaning of $\left[\bar{a}\ \bar{b}\ \bar{c}\right]$ is volume of parallelepiped with coter minus edges $\bar{a}$, $\bar{b}$ and $\bar{c}$.

We have,

$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = \left[\bar{b}\ \bar{c}\ \bar{a}\right] = \left[\bar{c}\ \bar{a}\ \bar{b}\right]$$

$$\left[\bar{a}\ \bar{b}\ \bar{c}\right] = -\left[\bar{b}\ \bar{a}\ \bar{c}\right]$$

Vector Triple Product:

Vector triple product of $\bar{a}$ $\bar{b}$ and $\bar{c}$ is cross product of $\bar{a}$ and $\left(\bar{b}\times\bar{c}\right)$ i.e. $\bar{a}\times\left(\bar{b}\times\bar{c}\right)$ or cross product of $\left(\bar{a}\times\bar{b}\right)$ and $\bar{c}$

$$\therefore\ \bar{a}\times\left(\bar{b}\times\bar{c}\right) = \left(\bar{a}\,.\,\bar{c}\right)\bar{b} - \left(\bar{a}\,.\,\bar{b}\right)\bar{c}$$

$$\left(\bar{a}\times\bar{b}\right)\times\bar{c} = \left(\bar{a}\,.\,\bar{c}\right)\bar{b} - \left(\bar{b}\,.\,\bar{c}\right)\bar{a}$$

Remark : Vector triple product is not associative in general

i.e. $\therefore\ \bar{a}\times\left(\bar{b}\times\bar{c}\right) \neq \left(\bar{a}\times\bar{b}\right)\times\bar{c}$

Coplanar Vectors:

Three vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are coplanar if $\left[\bar{a}\ \bar{b}\ \bar{c}\right] = 0$ for $\left|\bar{a}\right| \neq 0, \left|\bar{b}\right| \neq 0, \left|\bar{c}\right| \neq 0$

5.2 VECTORS DIFFERENTIATION

Let $\bar{v}$ be a vector function of a scalar t. Let $\partial\bar{v}$ be the small increment in a corresponding to the increment ∂t in t.

Then,

$$\partial \bar{v} = \bar{v}\left(t + \partial t\right) - \bar{v}(t)$$

$$\frac{\partial \bar{v}}{\partial t} = \frac{\bar{v}\left(t + \partial t\right) - \bar{v}(t)}{\partial t}$$

Taking limit $\partial t \longrightarrow 0$ we get,

$$\lim_{\partial t \to 0} \frac{\partial \bar{v}}{\partial t} = \lim_{\partial t \to 0} \frac{\bar{v}\left(t + \partial t\right) - \bar{v}(t)}{\partial t}$$

$$\frac{d\bar{v}}{dt} = \lim_{\partial t \to 0} \frac{\partial \bar{v}}{\partial t} = \lim_{\partial t \to 0} \frac{\bar{v}\left(t + \partial t\right) - \bar{v}(t)}{\partial t}$$

$$\frac{d\bar{v}}{dt} = \lim_{\partial t \to 0} \frac{\bar{v}\left(t + \partial t\right) - \bar{v}(t)}{\partial t}$$

Formulas of vector differentiation:

(i) $\frac{d}{dt} = \left(k\ \bar{v}\right) = k\frac{d\bar{v}}{dt}\left[\because\ k \text{ is a constant}\right]$

(ii) $\frac{d}{dt}\ \left(\bar{u} + \bar{v}\right) = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$

(iii) $\frac{d}{dt}\ \left(\bar{u}\ .\ \bar{v}\right) = \bar{u}\ .\ \frac{d\bar{v}}{dt} + \bar{v}\ .\frac{d\bar{u}}{dt}$

(iv) $\frac{d}{dt}\ \left(\bar{u} \times \bar{v}\right) = \bar{u} \times \frac{d\bar{v}}{dt} + \frac{d\bar{u}}{dt} \times \bar{v}$

(v) If $\bar{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Then, $\frac{d\bar{v}}{dt} = \frac{dv_1}{dt}\ \hat{i} + \frac{dv_2}{dt}\ \hat{j} + \frac{dv_3}{dt}\ \hat{k}$

Note:

If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then $r = \left|\bar{r}\right| = \sqrt{x^2 + y^2 + z^2}$

Example – 1:

If $\overline{r} = (t+1)\,\hat{i} + (t^2+t-1)\,\hat{j} + (t^2-t+1)\,\hat{k}$ find $\frac{d\overline{r}}{dt}$ and $\frac{d^2\overline{r}}{dt}$

Solution:–

$$\overline{r} = (t+1)\,\hat{i} + (t^2+t-1)\,\hat{j} + (t^2-t+1)\,\hat{k}$$

$$\frac{d\overline{r}}{dt} = \hat{i} + (2t+1)\,\hat{j} + (2t-1)\,\hat{k}$$

$$\frac{d^2\overline{r}}{dt^2} = 2\hat{j} + 2\,\hat{k}$$

Example – 2:

If $\overline{r} = \overline{a}\cos wt + \overline{b}\sin wt$ where w is constant show that

$\overline{r}\times\frac{d\overline{r}}{dt} = w\,(\overline{a}\times\overline{b})$ and $\frac{d^2\overline{r}}{dt^2} = -w\,\overline{r}$

Solution: –

$$\overline{r} = \overline{a}\cos wt + \overline{b}\sin wt\text{------------ (i)}$$

$$\frac{d\overline{r}}{dt} = \overline{a}\cos wt + \overline{b}\sin wt\text{------------ (ii)}$$

$$\therefore\ \overline{r}\times\frac{d\overline{r}}{dt} = (\overline{a}\cos wt + \overline{b}\sin wt)\times(-\overline{a}w\sin wt + \overline{b}w\cos wt)$$

$$= (\overline{a}\times\overline{b})\,w\cos^2 wt - (\overline{b}\times\overline{a})\,w\sin^2 wt \qquad \left[\begin{matrix}\because\ \overline{a}\times\overline{a} = \overline{0}\\ \overline{b}\times\overline{b} = \overline{0}\end{matrix}\right]$$

$$= (\overline{a}\times\overline{b})\,w\cos^2 wt + (\overline{a}\times\overline{b})\,w\sin^2 wt \qquad \left[\begin{matrix}\because\ \overline{b}\times\overline{a} = \overline{0}\\ = -\overline{a}\times\overline{b}\end{matrix}\right]$$

$$= (\overline{a}\times\overline{b})\,w\,[\cos^2 wt + \sin^2 wt]$$

$$= (\overline{a}\times\overline{b})\,w\,(1)$$

$$= w(\overline{a}\times\overline{b})$$

Again differentiating eq^n (ii) w.r.t. 't'

$$\frac{d^2\overline{r}}{dt^2} = -\overline{a}\,w^2\cos wt - \overline{b}\,w^2\sin wt$$

$$= -w^2\,(\overline{a}\cos wt + \overline{b}\sin wt)$$

$$= -w^2 r \text{ from (i)}$$

Example 3. Evaluate the following:

i) $\frac{d}{dt} = \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$ ii) $\frac{d}{dt} = \left[\bar{a} \quad \frac{d\bar{a}}{dt} \quad \frac{d^2\bar{a}}{dt^2}\right]$

Solution: – i) $\frac{d}{dt} = \left[\bar{a} \quad \bar{b} \quad \bar{c}\right]$

$$= \frac{d}{dt}\left[\bar{a} \,.\, \left(\bar{b} \times \bar{c}\right)\right]$$

$$= \bar{a} \,.\, \frac{d}{dt}\left(\bar{b} \times \bar{c}\right) + \left(\bar{b} \times \bar{c}\right) .\, \frac{d\bar{a}}{dt}$$

$$= \bar{a} \,.\left(\bar{b} \times \frac{d\bar{c}}{dt} + \frac{d\bar{b}}{dt} \times c\right) + \left(\bar{b} \times \bar{c}\right) .\, \frac{d\bar{a}}{dt}$$

$$= \bar{a} \,.\left(\bar{b} \times \frac{d\bar{c}}{dt}\right) + \bar{a} \,.\left(\frac{d\bar{b}}{dt} \times \bar{c}\right) + \left(\bar{b} \times \bar{c}\right) .\, \frac{d\bar{a}}{dt}$$

$$= \left[\bar{a} \;\; \bar{b} \;\; \frac{d\bar{c}}{dt}\right] + \left[\bar{a} \;\; \frac{d\bar{b}}{dt} \;\; \bar{c}\right] + \left[\bar{b} \;\; \bar{c} \;\; \frac{d\bar{a}}{dt}\right]$$

Solution: – ii) $\frac{d}{dt} = \left[\bar{a} \quad \frac{d\bar{a}}{dt} \quad \frac{d^2\bar{a}}{dt^2}\right]$

$$= \left[\bar{a} \;\; \frac{d\bar{a}}{dt} \;\; \frac{d^3\bar{a}}{dt^3}\right] + \left[\bar{a} \;\; \bar{c} \;\; \frac{d^2\bar{a}}{dt^2} \;\; \frac{d^2\bar{a}}{dt^2}\right] + \left[\frac{d\bar{a}}{dt} \;\; \frac{d^2\bar{a}}{dt^2} \;\; \frac{d\bar{a}}{dt}\right]$$

(From Result i)

$$= \left[\bar{a} \;\; \frac{d\bar{a}}{dt} \;\; \frac{d^3\bar{a}}{dt^3}\right] + 0 + 0$$

$$= \left[\bar{a} \,.\; \frac{d\bar{a}}{dt} \;\; \frac{d^3\bar{a}}{dt^3}\right]$$

Example 4. Evaluate the following: $\frac{d}{dt} = \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right]$

Solution: – iv) $\frac{d}{dt} = \left[\left(\bar{a} \times \bar{b}\right) \times \bar{c}\right]$

$$= \left(\bar{a} \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \frac{d}{dt}\left(\bar{a} \times \bar{b}\right) \times c$$

$$= \left(\bar{a} \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \left(\bar{a} \times \frac{d\bar{b}}{dt} + \frac{d\bar{a}}{dt} \times \bar{b}\right) \times c$$

$$= \left(\bar{a} \times \bar{b}\right) \times \frac{d\bar{c}}{dt} + \left(\bar{a} \times \frac{d\bar{b}}{dt}\right) \times c + \left(\frac{d\bar{a}}{dt} \times \bar{b}\right) \times c$$

Example 5. Show that $\hat{r} \times \frac{d\hat{r}}{dt} = \frac{\hat{r} \times \frac{d\hat{r}}{dt}}{r^2}$, where $\hat{r} = \frac{\overline{r}}{r}$

Solution : We have $\hat{r} = \frac{\overline{r}}{r}$

$$\therefore \quad \frac{d\hat{r}}{dt} = \frac{d}{dt}\left(\frac{\overline{r}}{r}\right)$$

$$= \frac{r\frac{d\overline{r}}{dt} - \overline{r}\frac{dr}{dt}}{r^2}$$

$$= \frac{1}{r}\frac{d\overline{r}}{dt} - \frac{r}{r^2}\frac{dr}{dt}$$

L.H.S. $\hat{r} = \frac{\overline{r}}{r}$

$$= \frac{\overline{r}}{r} \times \left(\frac{1}{r}\frac{d\overline{r}}{dt} - \frac{\overline{r}}{r^2}\frac{dr}{dt}\right)$$

$$= \frac{\overline{r}}{r} \times \frac{1}{r}\frac{d\overline{r}}{dt} - \frac{\overline{r}\times\overline{r}}{r^2}\frac{dr}{dt}$$

$$= \frac{\overline{r}}{r^2} \times \frac{dr}{dt} - \overline{0} \qquad [\because \overline{r}\times\overline{r} = 0]$$

$$= \frac{r \times \frac{d\overline{r}}{dt}}{r^2}$$

= R.H.S

Example 6. If $\overline{r} = t^3\, i + \left(2t^3 - \frac{1}{5t^2}\right) j$. Then show that $\overline{r} \times \frac{d\overline{r}}{dt} = \hat{k}$

Solution:

$$\overline{r} = t^3 i + \left(2t^3 - \frac{1}{5t^2}\right) j$$

$$\frac{d\overline{r}}{dt} = 3t^2 i + \left(6t^2 + \frac{2}{5t^3}\right) j$$

L.H.S.

$$r \times \frac{d\overline{r}}{dt} = \begin{vmatrix} i & j & k \\ t^3 & 2t^3 - \frac{1}{5t^2} & 0 \\ 3t^2 & 6t^2 + \frac{2}{5t^3} & 0 \end{vmatrix}$$

$$= i\,(0) - j(0) + k\left[t^3\left(6t^2 + \frac{2}{5t^3}\right) - 3t^2\left(2t^3 - \frac{1}{5t^2}\right)\right]$$

$$= k\left[t^3\left(6t^5 + \frac{2}{5} - 6t^5 + \frac{3}{5}\right)\right]$$

$$= \hat{k}$$

$$= \text{R. H. S.}$$

Example 7. If $\overline{r} = \overline{a}\,e^{mt} + \overline{b}\,e^{-mt}$. Show that $\frac{d^2\overline{r}}{dt^2} = n^2\overline{r}$

Solution:

$\overline{r} = \overline{a}\,e^{mt} + \overline{b}\,e^{-mt}$ ....(i)

$$\frac{d\overline{r}}{dt} = m\,\overline{a}\,e^{mt} - m\,\overline{b}\,e^{-mt}$$

$$\frac{d^2\overline{r}}{dt^2} = m^2\,\overline{a}\,e^{mt} + m^2\,\overline{b}\,e^{-mt}$$

$$= m^2\left(\overline{a}\,e^{mt} + \overline{b}\,e^{-mt}\right)$$

$$= m^2\,\overline{r} \qquad \text{(from (i))}$$

$$\frac{d^2\overline{r}}{dt^2} = m^2\,\overline{r}$$

Check your progress:

(1) If $\frac{d\overline{u}}{dt} = \overline{w}\times\overline{u}$ and $\frac{d\overline{v}}{dt} = \overline{w}\times\overline{v}$

Show that $\frac{d}{dt}\left(\overline{u}\times v\right) = \overline{w}\times\left(\overline{u}\times v\right)$

(2) If $\overline{r} = t^2 i + \left(3t^3 - t^2\right) j + \left(7t + 1\right)\hat{k}$ Find $\frac{d\overline{r}}{dt}, \frac{d^2\overline{r}}{dt^2}$

(3) If: $\overline{r} = t\,\hat{i} - t\,\hat{j} + \left(st - 1\right)\hat{k}$, Find $\frac{d\overline{r}}{dt}, \frac{d^2\overline{r}}{dt^2}, \left|\frac{d\overline{r}}{dt}\right|, \left|\frac{d^2\overline{r}}{dt^2}\right|$

(4) If $\overline{r} = \overline{e}^{t}\,i + \left(2\cos 3t\right) j + \left(7\sin 3t\right)\hat{j}$ Find $\frac{d^2\overline{r}}{dt^2}$ at $t = \frac{\pi}{2}$

(5) Show that: $\overline{a}\,.\,\frac{d\overline{a}}{dt} = a\,\frac{da}{dt}$ where $a = a_1 i + a_2 j + a_3 k$ and a is magnitude of $\overline{a}$.

5.3 VECTOR OPERATOR

The vector differential operator ∇ is defined as $\nabla = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$.

5.3.1 Gradient:

The gradient of a scalar function is denoted by grad ϕ or $\nabla\phi$ and is defined as $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}$. Note that grad ϕ is a vector quantity.

5.3.2 Geometric meaning of gradient:

The grad ϕ is a vector right angled to the surface, whose equation is $\phi\,(x, y,\ z) = c,$ where c is constant.

Hence for $\overline{r} = x\,i\ + y\,j\ + z\,k$ any point on surface $\therefore\ \nabla\phi\ .\ d\overline{r} = 0$

i.e. $\nabla\phi$ at is right angles to $d\overline{r}$ and $d\overline{r}$ lies on the tangent plane to the surface at $P(\overline{r})$.

$\therefore\ \nabla\phi\ \perp\ d\overline{r}$

Geometrically $\nabla\phi$ represents a vector normal to the surface $\phi\,(x,\ y, z) =$ constant.

Example 8: Find grad ϕ, where $\phi = x^2\ y^3\ e^z$

Solution: grad $\phi\ = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\left(x^2\ y^3\ e^z\right)$

$$= \hat{i}\,\frac{\partial}{\partial x}\left(x^2\ y^3\ e^z\right) + \hat{j}\,\frac{\partial}{\partial y}\left(x^2\ y^3\ e^z\right) + \hat{k}\,\frac{\partial}{\partial z}\left(x^2\ y^3\ e^z\right)$$

$$= \hat{i}\left(2xy^3e^z\right) + \hat{j}\left(3x^2\ y^2\ e^z\right) + \hat{k}\left(x^2\ y^3\ e^z\right)$$

$$= x\ y^2e^z\ \left(2y\ \hat{i} + 3x\hat{j} + xy\hat{k}\right)$$

Example 9: If $\overline{r} = x\hat{i} + y\hat{j}\ +\ z\hat{k}$ find grad r

Solution:

$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$r = \sqrt{x^2 + y^2 + z^2}$

$$\text{Grad } r = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\sqrt{x^2 + y^2 + z^2}$$

$$= \hat{i}\,\frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} + \hat{j}\,\frac{\partial}{\partial y}\sqrt{x^2 + y^2 + z^2} + \hat{k}\,\frac{\partial}{\partial z}\sqrt{x^2 + y^2 + z^2}$$

$$= \hat{i}\,\frac{1}{2\sqrt{x^2 + y^2 + z^2}}(2x) + \hat{j}\,\frac{1}{2\sqrt{x^2 + y^2 + z^2}}.(2y) + \hat{k}\,\frac{1}{2\sqrt{x^2 + y^2 + z^2}}(2z)$$

$$= \frac{x}{r}\,\hat{i} + \frac{y}{r}\,\hat{j} + \frac{z}{r}\,\hat{k}$$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$$\therefore \text{ grad } r = \frac{\bar{r}}{r}$$

Example 10: If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ find grad $\frac{1}{r}$

Solution:

$$\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore r^2 = x^2 + y^2 + z^2$$

$$2r\,\frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{2r}{2x} = \frac{x}{r},\ \frac{2r}{2y} = \frac{y}{r},\ \frac{2z}{2r} = \frac{z}{r}$$

$$\text{grad } \frac{1}{r} = \left(\hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \hat{j}\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \hat{k}\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$$

$$= \hat{i}\left(\frac{-1}{r^2}\,\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{-1}{r^2}\,\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{-1}{r^2}\,\frac{\partial r}{\partial z}\right)$$

$$= \frac{-1}{r^2}\left[i\frac{\partial r}{\partial x} + \hat{j}\,\frac{\partial r}{\partial y} + \hat{k}\,\frac{\partial r}{\partial y}\right]$$

$$= \frac{-1}{r^2}\left(\hat{i}\ \frac{x}{r} + \hat{j}\ \frac{y}{r} + \hat{k}\ \frac{z}{r}\right)$$

$$= \frac{-1}{r^2}.\ \frac{-1}{r}\ \left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$= \frac{-1}{r^3}\ r$$

$$= \frac{-r}{r^3}$$

Example 11: If $\phi = 2x^3y - y^2z$ find grad ϕ at (1, -1, 2)

Solution:

$$\text{grad } \phi = \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right)\left(2x^3y - y^2\ z\right)$$

$$= \hat{i}\ \frac{\partial}{\partial x}\ \left(2x^3y - y^2z\right) + \hat{j}\ \frac{\partial}{\partial y}\ \left(2x^3y - y^2\ z\right) + \hat{k}\ \frac{\partial}{\partial z}\ \left(2x^3y - y^2\ z\right)$$

$$= \hat{i}\ \left(6x^2y\right) + \hat{j}\ \left(2x^3 - 2yz\right) + \hat{k}\ \left(-y^2\right)$$

$$= \hat{i}\ 6x^2y + \hat{j}\ \left(2x^3 - 2yz\right) - \hat{k}\ y^2$$

At (1, -1, and 2)

$$\text{grad } \phi = 6\ (1)^2(-1)\ i + j\ \left(2(1)^3 - 2(-1)(2)\right) - \hat{k}\ (-1)^2$$

$$= 6\ i + j\ (2+4) - \hat{k}$$

$$= -6\ i + 6j\ - \hat{k}$$

Example 12: Evaluate grad e^{r^2}, where $r^2 = x^2 + y^2 + z^2$

Solution : Grad $\left(e^{r^2}\right) = \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right) e^{r^2}$

$$= \hat{i}\ \frac{\partial}{\partial x}\ \left(e^{r^2}\right) + \hat{j}\ \frac{\partial}{\partial y}\ \left(e^{r^2}\right) + \hat{k}\ \frac{\partial}{\partial z}\ \left(e^{r^2}\right)$$

$$= \hat{i}\ e^{r^2}.\partial r\frac{\partial r}{\partial x} + \hat{j}\ e^{r^2}.\partial r\frac{\partial r}{\partial y} + \hat{k}\ e^{r^2}.\partial r\frac{\partial r}{\partial z}$$

$$= \hat{i}\ e^{r^2}.\partial r.\frac{x}{r} + \hat{j}\ e^{r^2}.\partial r\frac{y}{r} + \hat{k}\ e^{r^2}.\partial r\frac{z}{r}$$

$$= r\ e^{r^2}\left(x\hat{i} + y\ \hat{j} + z\ \hat{k}\right)$$

$$= r\ e^{r^2}\overline{r}$$

Example 13: Find grad r^n

Solution: grad $r^n = \nabla r^n$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right) r^n$$

$$= \hat{i}\frac{\partial}{\partial x} r^n + \hat{j}\frac{\partial}{\partial y} r^n + \hat{k}\frac{\partial}{\partial z} r^n$$

$$= \hat{i}\, n\, r^{n-1}\frac{\partial r}{\partial x} + \hat{j}\, n\, r^{n-1}\frac{\partial r}{\partial y} + \hat{k}\, n\, r^{n-1}\frac{\partial r}{\partial z}$$

$$= \hat{i}\, n\, r^{n-1}\frac{x}{r} + \hat{j}\, n\, r^{n-1}\frac{y}{r} + \hat{k}\, n\, r^{n-1}\frac{z}{r}$$

$$= \hat{i}\, n\, r^{n-2}x + \hat{j}\, n\, r^{n-2}y + \hat{k}\, n\, r^{n-2}z$$

$$= n\, r^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= n\, r^{n-2} r$$

Example 14: Find grad $\log\left(x^2 + y^2 + z^2\right)$

Solution:

$$\text{grad}\ \log\left(x^2 + y^2 + z^2\right) = \text{grad}\ \log r^2 = \text{grad}\,(2\log r) = 2\ \text{grad}\,(\log r)$$

$$= 2\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)(\log r)$$

$$= 2\left(\hat{i}\frac{\partial}{\partial x}(\log r) + \hat{j}\frac{\partial}{\partial y}(\log r) + \hat{k}\frac{\partial}{\partial z}(\log r) \right)$$

$$= 2\left(\hat{i}\frac{1}{r}\frac{\partial r}{\partial x} + \hat{j}\frac{1}{r}\frac{\partial r}{\partial y} + \hat{k}\frac{1}{r}\frac{\partial r}{\partial z} \right)$$

$$= 2\left(\hat{i}\frac{1}{r}\frac{x}{r} + \hat{j}\frac{1}{r}\frac{y}{r} + \hat{k}\frac{1}{r}\frac{z}{r} \right)$$

$$= \frac{2}{r^2}\left(x\hat{i} + y\hat{j}\frac{1}{r}\frac{y}{r} + z\hat{k} \right)$$

$$= \frac{2\bar{r}}{r^2}$$

Example 15: Show that grad $\left(\frac{\bar{a}.\bar{r}}{r^n}\right) = \frac{\bar{a}}{r^n} = \frac{n\left(\bar{a}.\bar{r}\right)}{r^{n+2}}\, r$ where $\bar{r} = r\,i + yj + zk$

Solution: let

$$\overline{a} = a_1\ i + a_2\ j + a_3\ k$$

$$\therefore\ \overline{a}\ .\overline{r} = a_1\ x + a_2\ y + a_3$$

$$\therefore\ \text{grad}\ \left(\frac{\overline{a}.\ \overline{r}}{r^n}\right)$$

$$= \nabla\ \left(\frac{\overline{a}.\ \overline{r}}{r^n}\right)$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\ \left(\frac{a_1\ x + a_2\ y + a_3\ z}{r^n}\right)$$

$$\text{now}\ \therefore\ i\frac{\partial}{\partial x}\ \left(\frac{a_1\ x + a_2\ y + a_3\ z}{r^n}\right)$$

$$=\ i\ \left(\frac{r^n\ a_1 - (a_1\ x + a_2\ y + a_3\ z)\ nr^{n-1}\ \frac{\partial r}{\partial x}}{r^{2n}}\right)$$

$$=\ i\ \left(\frac{r^n\ a_1 - (a_1\ x + a_2\ y + a_3\ z)\ nr^{n-1}\ \frac{x}{r}}{r^{2n}}\right)$$

$$=\ i\ \left(\frac{r^n\ a_1 - (a_1\ x + a_2\ y + a_3\ z)\ nx^{n-1}\ r^{n-2}}{r^{2n}}\right)$$

similarly

$$=\ \hat{j}\ \frac{\partial}{\partial y}\ \left(\frac{(a_1\ x + a_2\ y + a_3\ z)}{r^n}\right)$$

$$=\ \hat{j}\ \left(\frac{r^n\ a_2 - (a_1\ x + a_2\ y + a_3\ z)\ ny\, r^{n-2}}{r^{2n}}\right)$$

and

$$=\ \hat{k}\ \frac{\partial}{\partial z}\ \left(\frac{(a_1\ x + a_2\ y + a_3\ z)}{r^n}\right)$$

$$=\ \hat{k}\ \left(\frac{r^n\ a_3 - (a_1\ x + a_2\ y + a_3\ z)\ nz\, r^{n-2}}{r^{2n}}\right)$$

$$\therefore\ \text{grad}\ \left(\frac{\overline{a}\ .\ \overline{r}}{r^n}\right)$$

$$= \frac{r^n\ \left(a_1\ \hat{i} + a_2\ \hat{j} + a_3\ \hat{k}\right) - (a_1\ x + a_2\ y + a_3\ z)\ nz\, r^{n-2}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)}{r^{2n}}$$

$$= \frac{\bar{a}r^n - n\,r^{n-2}\,\bar{r}\,(\bar{a}\,.\bar{r})}{r^{2n}}$$

$$= \frac{\bar{a}r^n}{r^{2n}} - \frac{-n\,(\bar{a}\,.\bar{r})\,r^{n-2}\,\bar{r}}{r^{2n}}$$

$$= \frac{\bar{a}r^n}{r^{2n}} - \frac{n\,(\bar{a}\,.\bar{r})}{r^{n+2}}\,\bar{r}$$

$$= \frac{\bar{a}}{r^n} - \frac{n\,(\bar{a}\,.\bar{r})}{r^{n+2}}\,\bar{r}$$

Check your progress:

(1) If $\bar{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $\bar{r} = |\bar{r}|$

Show that:

a) $\text{grad}\,(\log\ r) = \frac{\bar{r}}{r^2}$

b) $\text{grad}\ r^3 = 3\,r\,\bar{r}$

c) $\text{grad}\ f\,(r) = f^1\,(r)\,\frac{\bar{r}}{r}$

(2) If $\phi = 4x^2yz + 3xyz^2 - 5xyz$

Find grad ϕ at (3, 2, -1)

(3) Show that $\text{grad}\ r^3 = -3\,r^{-5}\,\bar{r}$

(4) If $F\,(x, y, z) = x^2 + y^2 + z^2$ Find ∇F at (1, 1, 1)

(5) Show that $\nabla f\,(r) \times \bar{r} = 0$ where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$

(6) Find unit vector normal to the surface $x^2 + y^2 + z^2 = 3a^2$ at (a, a, a)

[Hint :- Unit vector normal to surface ϕ i.e. $\frac{\nabla\phi}{|\nabla\phi|}$]

5.3.1 Divergence:

If v (x, y, z) = $v_1\hat{i} + v_2\,\hat{j} + v_3\,\hat{k}$ can be defined and differentiated at each point (x, y, z) in a region of space then divergence of v is defined as $\text{div}\ v = \nabla\ .\ \bar{v}$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . \left(v_1\ i + v_2\ j + v_3\ k \right)$$

$$= \frac{\partial}{\partial x}(v_1) + \frac{\partial}{\partial y}(v_2) + \frac{\partial}{\partial z}(v_3)$$

Example 16 If $\overline{F} = \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j} + \left(y^2 - 2xy\right)\hat{k}$, find $\overline{F}$

Solution: div $\overline{F} = \nabla . \overline{F}$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . \left\{ \left(x^2 - y^2\right)\hat{i} + 2xy\hat{j} + \left(y^2 - 2xy\right)\hat{k} \right\}$$

$$= \frac{\partial}{\partial x}\left(x^2 - y^2\right) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}\left(y^2 - 2xy\right)$$

$$= 2x + 2x + 0$$

$$= 4x$$

Example 17 Show that div $\overline{r} = 3$ where $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: div $\overline{r}$

$$= \nabla . \overline{F}$$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 1 + 1 + 1$$

$$= 3$$

Example 18 For $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ show that div $\left(r^n\ \overline{r}\right) = (n+3)\ r^n$ where $r = \left|\overline{r}\right|$

Solution: L.H.S. div $\left(r^n\ \overline{r}\right) = \nabla . \left(r^n\ r\right)$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) . r^n \left(x\hat{i} + y\hat{j} + z\hat{k} \right)$$

$$= \frac{\partial}{\partial x}\left(r^n\ x\right) + \frac{\partial}{\partial y}\left(r^n\ y\right) + \frac{\partial}{\partial z}\left(r^n\ z\right)$$

$$= r^n(1) + x\ nr^{n-1}\ \frac{\partial r}{\partial x} + r^n(1) + y\ nr^{n-1}\ \frac{\partial r}{\partial y} + r^n(1) + z\ nr^{n-1}\ \frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left(x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + + z\frac{\partial r}{\partial z}\right)$$

$$= 3r^n + nr^{n-1}\left(x.\frac{x}{r} + y.\frac{y}{r} + z.\frac{z}{r}\right)$$

$$= 3r^n + nr^{n-1}\frac{\left(x^2 + y^2 + z^2\right)}{r}$$

$$= 3r^n + nr^{n-1}\frac{r^2}{r}$$

$$= 3r^n + nr^n$$

$$= (3 + n)r^n$$

$$= \text{R.H.S.}$$

Example 19 Evaluate div $(\hat{r})$ where $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: We have $\hat{r}^n = \frac{\overline{r}}{r}$

$$= \frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}$$

$\therefore$ div $(\hat{r})$

$$= \nabla.\hat{r}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2} + \frac{r(1) - y\frac{\partial r}{\partial y}}{r^2} + \frac{r(1) - z\frac{\partial r}{\partial z}}{r^2}$$

$$= \frac{r - x\left(\frac{x}{r}\right)}{r^2} + \frac{r - y\frac{y}{r}}{r^2} + \frac{r - z\frac{z}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3}$$

$$= \frac{r^2 - x^2 + r^2 - y^2 + r^2 - z^2}{r^3}$$

$$= \frac{3r^2 - \left(x^2 + y^2 + z^2\right)}{r^3}$$

$$= \frac{3r^2 - r^2}{r^3}$$

$$= \frac{2}{r}$$

Example 20 If $F = x^2\ y^3\ z^4$ Find div (grad F)

Solution: grad F

$$= \nabla F$$

$$= \left(\hat{i}\ \frac{\partial}{\partial x} + \hat{j}\ \frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z} \right) \left(x^2\ y^3\ z^4 \right)$$

$$= 2xy^3z^4\ \hat{i} + 3y^2z^4\ \hat{j} + 4x^2y^3z^3\ \hat{k}$$

$\therefore$ div (grad F)

$$= \nabla.\left(2xy^3z^4\ \hat{i} + 3y^2x^2z^4\ \hat{j} + 4x^2y^3z^3\ \hat{k} \right)$$

$$= \frac{\partial}{\partial x}\left(2xy^3z^4 \right) + \frac{\partial}{\partial y}\left(3y^2\ x^2\ z^4 \right) + \frac{\partial}{\partial z}\left(4\ x^2y^3\ z^3 \right)$$

$$= 2xy^3z^4 + 6x^2y\ z^4 + 12\ x^2y^3\ z^2$$

Example 21 Find the value of div $\left(\bar{a} \times \bar{r} \right) r^n$ where $\bar{a}$ is a constant vector and $\bar{r} = x\hat{i} + y\ \hat{j} + z\ \hat{k}$

Solution: div $\left(\bar{a} \times \bar{r} \right) r^n$

$$= \hat{i}\ \frac{\partial}{\partial x} . \left\{ \left(\bar{a} \times \bar{r} \right) r^n \right\}$$

$$= \sum \hat{i} . \left[\left\{ \frac{\partial}{\partial x} . \left\{ \left(\bar{a} \times \bar{r} \right) \right\} \right\} r^n + \left(\bar{a} \times \bar{r} \right) \frac{\partial}{\partial x} \left(r^n \right) \right]$$

$$= \sum \hat{i} . \left[\left(\bar{a} \times \frac{\partial \bar{r}}{\partial x} \right) r^n + \left(\bar{a} \times \bar{r} \right) n\ r^{n-1}\ \frac{\partial r}{\partial x} \right]$$

$$\left[\because \frac{\partial \bar{a}}{\partial x} = 0 \right]$$

$$= \sum \hat{i} . \left[\left(\bar{a} \times \hat{i} \right) r^n + \left(\bar{a} \times \bar{r} \right) n\ r^{n-1}\ \frac{x}{r} \right]$$

$$= \sum \hat{i} . \left[\left(\bar{a} \times \hat{i} \right) r^n + nx\ r^{n-2}\ \left(\bar{a} \times \bar{r} \right) \right]$$

$$= \sum nr^{n-2} \left(x\ \hat{i} \right) \left(\bar{a} \times \bar{r} \right) \qquad \left[\because \hat{i}.\left(\bar{a} \times \hat{i} \right) = 0 \right]$$

$$= nr^{n-2} \left(\bar{a} \times \bar{r} \right) \sum x\ \hat{i}$$

$= nr^{n-2}(\bar{a} \times \bar{r})\ \bar{r}$ $\qquad\left[\because \sum x\hat{i} = \bar{r}\right]$

$= nr^{n-2}\left[(\bar{a} \times \bar{r}) \cdot \bar{r}\right]$

$= nr^{n-2}(0)$

$= 0$

5.3.4 Solenoidal Function: A vector function $\bar{F}$ is called solenoidal if $\operatorname{div} \bar{F} = 0$ at all points of the function.

5.3.5 Curl: The curl of a vector point function $\bar{F}$ is defined as curl $\bar{F} = \nabla \times \bar{F}$ if $F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$.

$$\because \text{curl } \bar{F} = \nabla \times \bar{F}$$

$$= (\nabla \times \bar{F})$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

The curl of the linear velocity of any particle of rigid body is equal to twice the angular velocity of body.

i.e. if $\bar{w} = w_1\hat{i} + w_2\hat{j} + w_3\hat{k}$ be the angular velocity of any particle of the body with position vector defined as $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ then linear velocity $\bar{v} = \bar{w} \times \bar{r}$.

Hence curl $\bar{v} = \nabla \times \bar{v}$

$$= \nabla \times (\bar{w} \times \bar{r})$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \nabla \times \left[\hat{i}\ (w_2 z - w_3 y) - \hat{j}\ (w_1 z - w_3 x) + \hat{k}\ (w_1 y - w_2 x)\right]$$

$$= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ w_2 z - w_3 y & w_3 x - w_1 z & w_1 y - w_2 x \end{vmatrix}$$

$$= \hat{i}\ (w_1 + w_1) - \hat{j}\ (-w_2 - w_2) + \hat{k}\ (w_3 + w_3)$$

$$= 2w_1\hat{i} + 2w_2\hat{j} + 2w_3\hat{k}$$

$$= 2\overline{w}$$

$$\therefore\ \text{curl}\ \overline{v} = 2\overline{w}$$

5.3.6 Irrotational field:

A vector point function $\overline{F}$ is called irrotational if $\overline{F} = \overline{0}$ at all points of the function.

Example 22 Find curl (curl $\overline{F}$) If $\overline{F} = x^2\ y\ \hat{i}\ -2\ x\ z\ \hat{j} + 2\ y\ z\ \hat{k}$ at (1, 0, 2)

Solution: Curl $\overline{F}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & -2xy & 2yz \end{vmatrix}$$

$$= (2z + 2x)\hat{i} + (-2z - x^2)\ \hat{k}$$

$$\therefore\ \text{curl curl}\ (\overline{F}) = \nabla \times \left[(2z + 2x)i + 0j + (-2z - x^2)\ \hat{k}\right]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z + 2x & 0 & -2z - x^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(-2z - x^2) - \frac{\partial}{\partial z}(0)\right] - \hat{j}\left[\frac{\partial}{\partial x}(-2z - x^2) - \frac{\partial}{\partial z}(2z + 2x)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(2z + 2x)\right]$$

$$= \hat{i}(0) - \hat{j}\ (-2x - 2) + \hat{k}(0)$$

$$= (2x + 2)\ \hat{j}$$

At (1, 0, 2)

$\left(\text{curl } \overline{F}\right) = \left[2(1) + 2\right]\hat{j}$

$= 4\hat{j}$

Example 23 Find curl $\overline{V}$ if $\overline{V} = \left(x^2 + yz\right)\hat{i} + \left(y^2+2x\right)\hat{j}\left(z^2+xy\right)\hat{k}$

Solution: curl $\overline{V}$

$= \nabla \times \overline{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+yz & y^2+zx & z^2+xy \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}\left(z^2+xy\right) - \frac{\partial}{\partial z}\left(y^2+2x\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(z^2+xy\right) - \frac{\partial}{\partial z}\left(y^2+yz\right)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\left(y^2+2x\right) - \frac{\partial}{\partial y}\left(x^2+yz\right)\right]$$

$= \hat{i}\left(x - x\right) - \hat{j}\left(y - y\right) + \hat{k}\left(z - z\right)$

$= \overline{0}$

Example 24 Evaluate curl $\overline{r}$ where if $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution: Curl $\overline{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) - \hat{j}\left(\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)$$

$= 0\hat{i} - 0\hat{j} + 0\hat{k}$

$= \overline{0}$

Example 25 Evaluate curl $\left(\frac{\hat{r}}{r}\right)$ where if $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Solution:

$\hat{r} = \left(\frac{\overline{r}}{r^2}\right)$

$\therefore \frac{\hat{r}}{r} = \frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}$

$\therefore \text{curl}\left(\frac{\hat{r}}{r}\right) = \nabla \times \left(\frac{x}{r^2}\hat{i} + \frac{y}{r^2}\hat{j} + \frac{z}{r^2}\hat{k}\right)$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix}$

$= \hat{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right] - \hat{j}\left[\frac{\partial}{\partial x}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{x}{r^2}\right)\right]$

$+ \hat{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r^2}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^2}\right)\right]$

$= \hat{i}\left[\frac{-2z}{r^3}\frac{2r}{2y} + \frac{2y}{r^3}\frac{2r}{2z}\right]$ + -------------+---------------

$= \hat{i}\left[\frac{-2z}{r^3}\frac{y}{r} + \frac{2y}{r^3}\frac{z}{r}\right]$ + -------------+---------------

$= \hat{i}\left[\left(\frac{2yz - 2yz}{r^3}\right) + \hat{j}\left(\frac{2zx - 2zx}{r^3}\right) + \hat{k}\left(\frac{2xy - 2xy}{r^3}\right)\right]$

$= 0\hat{i} + 0\hat{j} + 0\hat{k}$

$= \overline{0}$

Example 25 If $\overline{F} = x^2y\,\hat{i} + xz\hat{j} + 2yz\hat{k}$ find div (curl $\overline{F}$)

Solution: curl $\overline{F}$

$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & xz & 2yz \end{vmatrix}$

$= \hat{i}\left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(xz)\right] - \hat{j}\left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 z)\right]$

$+ \hat{k}\left[\frac{\partial}{\partial x}(x\,z) - \frac{\partial}{\partial y}(x^2 z)\right]$

$= \hat{i}\,(2z - x) - \hat{j}(0 - 0) + \hat{k}\,(z - x^2)$

$= (2z - x)\,\hat{i} + (z - x^2)\hat{k}$

div (curl $\overline{F}$)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) . \left[(2z - x)\ \hat{i} + \left(z - x^2\right)\ \hat{k}\right]$$

$$= \frac{\partial}{\partial x}\ (2z - x) + \frac{\partial}{\partial z}\ \left(z - x^2\right)$$

$$= -1 + 1$$

$$= 0$$

Example 27 If $\overline{F} = \text{grad}\ (xy + yz + zx)$, find (curl $\overline{F}$).

Solution: $\overline{F} = \text{grad}\ (xy + yz + zx)$

$$= \nabla\ (xy + yz + zx)$$

$$== \left[i\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]\ (xy + yz + zx)$$

$$= \hat{i}\ \frac{\partial}{\partial x}\ (xy + yz + zx) + \hat{j}\ \frac{\partial}{\partial y}\ (xy + yz + zx) + \hat{k}\frac{\partial}{\partial z}(xy + yz + zx)$$

$$= \hat{i}\ (y + z) + \hat{j}(x + z) + \hat{k}\ (y + x)$$

$\therefore$ (curl $\overline{F}$)

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + z & x + z & x + y \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x + y) - \frac{\partial}{\partial z}\ (x + z)\right] - \hat{j}\left[\frac{\partial}{\partial x}\ (x + y) - \frac{\partial}{\partial z}\ (y + z)\right]$$

$$+ \hat{k}\left[\frac{\partial}{\partial x}\ (x + z) - \frac{\partial}{\partial y}\ (y + z)\right]$$

$$= \hat{i}(1 - 1) - \hat{j}\ (1 - 1) + \hat{k}\ (1 - 1)$$

$$= 0\ \hat{i} + 0\ \hat{j} + 0\ \hat{k}$$

$$= \overline{0}$$

5.4 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

(i) $\nabla\ (f \pm g) = \nabla f \pm \nabla g$

(ii) $\nabla . \left(\overline{A} \pm \overline{B}\right) = \nabla . \overline{A} \pm \nabla . \overline{B}$

(iii) $\nabla \times \left(\overline{A} \pm \overline{B}\right) = \nabla \times \overline{A} \pm \nabla . \overline{B}$

Proof:

(**i**) $\nabla\ (f \pm g) = \hat{i}\left(\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\ \frac{\partial}{\partial z}\right)(f\ \pm\ g)$

$= \hat{i}\ \frac{\partial}{\partial x}\ (f\ \pm\ g)\ \pm\ \hat{j}\ \frac{\partial}{\partial y}\ (f\ \pm\ g) + \hat{k}\ \frac{\partial}{\partial z}\ (f\ \pm\ g)$

$= \left(\ \hat{i}\frac{\partial}{\partial x}f + \hat{j}\frac{\partial}{\partial y}f + \hat{k}\ \frac{\partial}{\partial z}\ \right) \pm \left(\ \hat{i}\frac{\partial}{\partial x}g + \hat{j}\frac{\partial}{\partial y}g + \frac{\partial}{\partial z}g\ \right)$

$= \nabla f\ \pm\ \nabla g$

(**ii**) Let $\bar{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$

$\bar{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$

$\therefore\ \nabla\ .\ (\bar{A} \pm \bar{B})$

$=\ \nabla\ .\ \left[(\bar{A}_1 \pm \bar{B}_1)\ \hat{i} + (\bar{A}_2 \pm \bar{B}_2)\ \hat{j} + (\bar{A}_3 \pm \bar{B}_3)\ \hat{k}\right]$

$= \frac{\partial}{\partial x}(\bar{A}_1 \pm \bar{B}_1) + \frac{\partial}{\partial y}\ (\bar{A}_2 \pm \bar{B}_2) + \frac{\partial}{\partial z}\ (\bar{A}_3 \pm \bar{B}_3)$

$= \frac{\partial}{\partial x}(A_1) + \frac{\partial}{\partial y}\ (A_2) + \frac{\partial}{\partial z}\ (A_3)\ \pm\ \left[\frac{\partial}{\partial x}(B_1) + \frac{\partial}{\partial y}\ (B_2) + \frac{\partial}{\partial z}\ (B_3)\right]$

$= \nabla\ .\ \bar{A}\ \pm \nabla\ .\ \bar{B}$

(**ii**) Let

$\nabla\ \times\ (\bar{A} \pm \bar{B})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \bar{A}_1 \pm \bar{B}_1 & \bar{A}_2 \pm \bar{B}_2 & \bar{A}_3 \pm \bar{B}_3 \end{vmatrix}$$

$=\ \sum \hat{i}\left[\frac{\partial}{\partial y}(\bar{A}_3 \pm \bar{B}_3) - \frac{\partial}{\partial z}(\bar{A}_2 \pm \bar{B}_2)\right]$

$=\ \sum \hat{i} \times\ \frac{\partial}{\partial x}\ (\bar{A} \pm \bar{B})$

$=\ \sum \hat{i} \times \left(\frac{\partial \bar{A}}{\partial x}\ \pm\ \frac{\partial \bar{B}}{\partial x}\right)$

$=\ \sum \hat{i} \times\ \frac{\partial \bar{A}}{\partial x}\ \pm\ \sum \hat{i}\ \times\ \frac{\partial \bar{B}}{\partial x}$

$=\ \nabla \times \bar{A} \pm \nabla \times \bar{B}$

Check your progress:

(**1**) If $\overline{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, $\overline{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ Evaluate div $\left(\overline{A}\times\overline{r}\right)$

(**2**) Prove that

$$\text{div}\left(\frac{\log r}{r}\overline{r}\right) = \frac{1}{r}\left(1 + 2\log r\right)$$

(**3**) For $\overline{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$) show that the vector div $\left(\frac{\overline{r}}{r^3}\right)$ is both solenoidal and irrotational.

(**4**) Prove that div $\left(\overline{a}.\,\overline{r}\right)\,\overline{a} = \left|\overline{a}\right|^2$

(**5**) For $\overline{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ show that $\nabla.\left(\nabla r^n\right) = n\,\left(n+1\right)r^{n-2}$

(**6**) show that the vector $\overline{F} = yz\hat{i} + zx\hat{j} + xy\hat{k}$ solenoidal.

(**7**) If $\overline{A} = \left(ax + 3y + 4z\right)\hat{i} + \left(x - 2y + 3z\right)\hat{j} + \left(3x + 2y - z\right)\hat{k}$ is solenoidal find value of a.

(**7**) Find the direction derivative of a scalar field $\phi = x^2\,y\,z$ at (4, -1, 2) in the direction of (3, 2, 1).

[**Hint :-** direction derivative of ϕ (x, y, z) along $\overline{a}$ is = $\overline{a}$. grad ϕ]

5.4 PROPERTIES OF GRADIENT, DIVERGENCE AND CURL

1) If $\overline{S}$ represents displacement vector, $\frac{d\overline{s}}{dt}$ represents velocity and $\frac{d^2\overline{s}}{dt^2}$ represents acceleration.

2) For $\frac{d\overline{s}}{dt}\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$

$$\text{grad } f = \nabla F$$

$$\text{grad } \overline{F} = \nabla.\,\overline{F}$$

$$\text{curl } \overline{F} = \nabla \times \overline{F}$$

3) grad F and curl $\overline{F}$ are vector quantities.

4) div $\overline{F}$ is scalar quantity.

Using equation (1), we have

$$\frac{dt}{dx}-4=t^2$$

$$\therefore \frac{dt}{dx}=t^2+4$$

$$\therefore \frac{1}{t^2+4}\cdot dt=dx$$

Which is invariable seperable form

Integrating we get,

$$\therefore \int\frac{1}{t^2+4}\cdot dt=\int dx+\text{constant}$$

$$\therefore \int\frac{1}{2}\cdot \tan^{-1}\left(\frac{t}{2}\right)=x+c$$

$$t=x+y$$

$$\therefore \frac{1}{2}\cdot\tan^{-1}\left(\frac{x+y}{2}\right)=x+c$$

$$\therefore \tan^{-1}\left(\frac{x+y}{2}\right)=2x+c_1 \text{ where } c_1=c$$

Which is the required general solution

4) $$(x+y)\cdot\frac{dy}{dx}+y=0 ----------(1)$$

Put $x+y=t$

Differentiating with respect to x, we get

$$\therefore 1+\frac{dy}{dx}=\frac{dt}{dx}$$

$$\therefore \frac{dy}{dx}=\frac{dt}{dx}-1$$

Using equation (1), we have

$$\therefore t\cdot\left(\frac{dt}{dx}-1\right)+t-x=0$$

$$\frac{dt}{dx}-1=\frac{x-t}{t}$$

$$\therefore \frac{dt}{dx}-1=\frac{x}{t}-1$$

$$\therefore \frac{dt}{dx}=\frac{x}{t}$$

$$xdx=tdt$$

Which is invariable seperable form

Integrating we get, $\int xdx=\int tdt+\text{constant}$

$$\frac{x^2}{2}=\frac{t^2}{2}+c$$

$$\therefore x^2=t^2+2c$$

$$t=x+y$$

$$\therefore x^2=(x+y)^2+2c$$

$$\therefore x^2=x^2+2xy+y^2+2c$$

$$\therefore 2xy+y^2=-2c$$

$$\therefore y^2+2xy=c_1 \text{ where } c_1=-2c$$

Which is the required general solution

5) Solve: $$\left(\frac{y}{x}\cos\frac{y}{x}\right)\cdot dx-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot dy=0$$

Soln:

The equation is, $$\left(\frac{y}{x}\cos\frac{y}{x}\right)\cdot dx-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot dy=0$$

Substitute $$\frac{y}{x}=v$$

$$\therefore y=vx$$

Differentiating above with respect to x, we get

$$\therefore \frac{dy}{dx}=v+x\cdot\frac{dv}{dx}$$

But the above equation can be written as

$$\therefore \frac{y}{x}\cdot\cos\frac{y}{x}-\left(\frac{x}{y}\cdot\sin\frac{y}{x}+\cos\frac{y}{x}\right)\cdot\frac{dy}{dx}=0$$

$$\therefore v\cos v-\left(\frac{1}{v}\cdot\sin v+\cos v\right)\cdot\left(v+x\cdot\frac{dy}{dx}\right)=0$$

By rearranging the terms, we have

$$\therefore \frac{1}{x}\cdot dx = -\frac{\sin v + v\cos v}{v\sin v}dv$$

$$\therefore \frac{1}{x}\cdot dx + \frac{\sin v + v\cos v}{v\sin v}dv = 0$$

Which is invariable seperable form

Integrating we get,

$$\therefore \int\frac{1}{x}\cdot dx + \int\frac{\sin v + v\cos v}{v\sin v}dv = \text{constant}$$

$$\therefore \log x + \log(v\sin v) = c$$

$$\log(x\cdot v\sin v) = \log c$$

$$xv\cdot \sin v = c$$

$$v = \frac{y}{x}$$

$$\therefore x\cdot\frac{y}{x}\sin\frac{y}{x} = c$$

$$\therefore y\sin\frac{y}{x} = c$$

Which is the required general solution

Checkpoint :-

Solve the following

1) $\frac{dy}{dx} + e^{y/x} = \frac{y}{x}$ Ans : $\log cx = e^{-y/x}$

2) $\left(1+e^{x/y}\right) + e^{x/y}\left(1-\frac{x}{y}\right)\cdot\frac{dy}{dx} = 0$ Ans : $x + y\cdot e^{x/y} = c$

3) $(2x-y)\cdot e^{y/x} + \left(y + x\cdot e^{y/x}\right)\cdot\frac{dy}{dx} = 0$ Ans : $y^2 + 2x^2e^{y/x} = c$

$$\left[\tan\frac{y}{x} - \frac{y}{x}\cdot\sec^2\frac{y}{x}\right]dx + \sec^2\frac{y}{x}\cdot dy = 0$$

Ans $x + \tan\left(\frac{y}{x}\right) = c$

Homogeneous Equations

A differential equation Mdx+Ndy=0 is said to be homogeneous if M & N are homogeneous functions of x and y of same degree

Working Rule :

1) Check whether differential equation is homogenous in x and y

2) Express $\frac{dy}{dx}$ in terms of x and y

3) Put y=vx

4) $\therefore \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$

5) Seperate x andy variables and get F(ex)dx+ g(v) dv=0

6) Solve by integration

7) Put $v = \frac{y}{x}$ and simplify

Solved examples:-

1) Solve : $\left(x^2+y^2\right)dx + 2xy\cdot dy = 0$

Soln We have : $\left(x^2+y^2\right)dx + 2xy\cdot dy = 0$

Here M and N are homogeneous expressions in x and y of the second degree

$$\therefore 2xy\cdot dy = -\left(x^2+y^2\right)dx$$

$$\therefore 2xy\cdot dy = -\left(x^2+y^2\right)dx$$

$$\therefore \frac{dy}{dx} = -\cdot\left(\frac{x^2+y^2}{2xy}\right) \text{------------} (1)$$

put y = vx

$$\therefore \frac{dy}{dx} = v + x\cdot\frac{dv}{dx}$$

Using equation 1 we have

$$v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{-2x\cdot vx}$$

$$\therefore v + x\frac{dv}{dx} = \frac{\cancel{x^2}\left(1+v^2\right)}{-2v\cdot\cancel{x^2}}$$

$$\therefore\ x\frac{dv}{dx}=\frac{1+2v^2}{-2v}-1$$

$$\therefore\ x\frac{dv}{dx}=\frac{1+3v^2}{-2v}$$

$$\therefore\ \frac{-2v}{1+3v^2}\cdot dv=\frac{1}{x}dx$$

Which is invariable seperable form

Integrating above expression we have

$$\therefore\ -\frac{1}{3}\int\frac{6v}{1+3v^2}\cdot dv=\int\frac{1}{x}\,dx+\text{constant}$$

$$\therefore\ -\frac{1}{3}\log\left(1+3v^2\right)=\log x+\log c$$

$$\therefore\ -\frac{1}{3}\log\left(1+3v^2\right)=\log(cx)$$

$$\therefore\ \log\left(1+3v^2\right)=-3\log(cx)$$

$$\therefore\ \log\left(1+3v^2\right)=-3\log(cx)^{-3}$$

$$\therefore\ 1+3v^2=\frac{1}{c^3x^3}$$

$$v=\frac{y}{x}$$

$$\therefore\ 1+3\cdot\frac{y^2}{x^2}=\frac{1}{c^3x^3}$$

$$\therefore\ x^3+3xy^2=\frac{1}{c^3}$$

$$\therefore\ x^3+3xy^2=k \text{ where } k=\frac{1}{c^3}$$

which is the required general solution

2) Solve $$y^2+x^2\cdot\frac{dy}{dx}=xy\cdot\frac{dy}{dx}$$

Soln :

The given equation is

$$y^2+x^2\cdot\frac{dy}{dx}=xy\cdot\frac{dy}{dx}$$

$$y^2+x^2\cdot\frac{dy}{dx}=xy\cdot\frac{dy}{dx}$$

$$\therefore\ y^2=xy\cdot\frac{dy}{dx}-x^2\cdot\frac{dy}{dx}$$

$$\therefore\ y^2=\frac{dy}{dx}\left(xy-x^2\right)$$

$$\therefore\ \frac{dy}{dx}=\frac{y^2}{xy-x^2}\ ---------(1)$$

Which is a homogeneous equation

Put $y=vx$

using equation (1),we have

$$\therefore\ v+x\cdot\frac{dv}{dx}=\frac{v^2x^2}{x\cdot vx-x^2}$$

$$\therefore\ v+x\cdot\frac{dv}{dx}=\frac{v^2x^2}{x^2(v-1)}$$

$$\therefore\ x\cdot\frac{dv}{dx}=\frac{v^2}{(v-1)}-v$$

$$\therefore\ x\cdot\frac{dv}{dx}=\frac{v}{(v-1)}$$

$$\therefore\ \frac{v-1}{v}\cdot dv=\frac{1}{x}\cdot dx$$

$$\therefore\ \left(1-\frac{1}{v}\right)dv=\frac{1}{x}dx$$

Which is invariable seperable form

Integrating we get,

$$\int\left(1-\frac{1}{v}\right)dv = \int\frac{1}{x}\,dx + \text{constant}$$

$$\therefore\ v\log v = \log x + \log c$$

$$\therefore\ v = \log v + \log x + \log c$$

$$\therefore\ v = \log(vxc)$$

$$v = \frac{y}{x}$$

$$\therefore\ \frac{y}{x} = \log\left(\frac{y}{x}\cdot x\cdot c\right)$$

$$\therefore\ \frac{y}{x} = \log cy$$

$$\therefore\ y = x\log cy$$

Which is the required general solution

3) solve : $(x^3+y^3)dx - 3xy^2\cdot dy = 0$

Which is a homogenous equation

$$(x^3+y^3)dx = 3xy^2\cdot dy$$

$$\therefore \frac{dy}{dx} = \frac{x^3+y^3}{3xy^2} \text{--------}(1)$$

Put $y = vx$

$$\therefore\ \frac{dy}{dx} = v + x\frac{dv}{dx}$$

using equation 1 we have

$$\therefore\ v + x\frac{dv}{dx} = \frac{x^3+v^3x^3}{3x\cdot v^2x^2}$$

$$\therefore\ v + x\frac{dv}{dx} = \frac{x^3(1+v^3)}{3v^2\cdot x^3}$$

$$\therefore\ x\frac{dv}{dx} = \frac{1+v^3}{3v^2} - v$$

$$\therefore\ x\frac{dv}{dx} = \frac{1-2v^3}{3v^2} -$$

$$\therefore\ \frac{3v^2}{1-2v^3}\cdot dv = \frac{1}{x}dx$$

Which is invariable seperable form

Integrating we have

$$-\frac{1}{2}\cdot\int\frac{6v^2}{2v^3-1}\cdot dv = \int\frac{1}{x}\,dx + \text{constant}$$

$$\therefore\ -\frac{1}{2}\log(2v^3-1) = \log x + \log c$$

$$\therefore\ \log(2v^3-1) = -2\log(cx)$$

$$\therefore\ \log(2v^3-1) = \log(cx)^{-2}$$

$$\therefore\ (2v^3-1) = \frac{1}{c^2x^2}$$

Put $v = \frac{y}{x}$

$$\therefore\ 2\frac{y^3}{x^3} - 1 = \frac{1}{c^2x^2}$$

$$\therefore\ 2y^3 - x^3 = \frac{x}{c^2}$$

$$\therefore\ 2y^3 - x^3 = kx \text{ where } k = \frac{1}{c^2}$$

Which is the required general solution

4) solve : $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\cdot dy = 0$

4) soln:

The given equation is ,

$$\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\cdot dy = 0$$

$$\therefore \frac{dy}{dx}=\frac{y\sec^2\frac{y}{x}-x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}}$$

$$\therefore \frac{dy}{dx}=\frac{y}{x}-\frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}} \text{ - - - - - - - - - (1)}$$

Which is a homogeneous equation

put $y=vx$

Using equation 1 we have

$$\therefore v+x\cdot\frac{dv}{dx}=v-\frac{\tan v}{\sec^2 v}$$

$$\therefore x\cdot\frac{dv}{dx}=\frac{\tan v}{\sec^2 v}$$

$$\therefore \frac{\sec^2 v}{\tan v}\cdot dv=-\frac{1}{x}\cdot dx$$

$$\therefore \frac{\sec^2 v}{\tan v}\cdot dv+-\frac{1}{x}\cdot dx=0$$

Which is invariable seperable form

Integrating we get,

$$\therefore \int\frac{\sec^2 v}{\tan^2 v}\cdot dv+\int\frac{1}{x}dx=\text{constant}$$

$$\therefore \log\tan v+\log x=\log c$$

$$\therefore \log(\tan v\cdot x)=\log c$$

$$\therefore x\cdot\tan v=c$$

Put $v=\frac{y}{x}$

$$\therefore x\cdot\tan\frac{y}{x}=c$$

Which is the required general solution

Check Point

1)solve the following

(i) $x\,dy-y\,dx=\sqrt{x^2+y^2}\cdot dx$

Ans: $y+\sqrt{x^2+y^2}=cx^2$

(ii) $\left(x+y\cdot\cot\frac{x}{y}\right)dy-y\,dx=0$

Ans: $y=c\sec\frac{x}{y}$

(iii) $y^2+x^2\cdot\frac{dy}{dx}=xy\cdot\frac{dy}{dx}$

Ans: $cy=e^{y/x}$

(iv) $(x^2-y^2)dx=2xy\,dy$

Ans: $x(x^2-3y^2)=c$

(v) $x\frac{dy}{dx}=y+\sqrt{x^2+a^2}$

Ans: $y=c\cdot e^{x^2/3y^2}$

(vi) $(x+y)\cdot\frac{dy}{dx}=x-y$

Ans: $-y^2-2xy+x^2=c$

Exact Differential Equation

Definition :-

The equation Mdx+Ndy=0 is said to be an exact differential equation if and only it.

Mdx+ Ndy=du

Where u is some function of x and y

e.g. xdy+ydx=0 is exact

$$\because\ u = xy$$

Where

xdy+ydy = du

Necessary and sufficient condition :-

The necessary and sufficient condition that the equation Mdx+Ndy=0 is exact is.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Rules for the General solution :-

If the equation Mdx+Ndy=0 is exact then its general solution is given by

$$\int M\ (\text{treat y as constant})\ dx + \int N\ (\text{terms free from x})\ dy = c$$

Where

(1) In first integral with respect to x, treat y as constant

(ii) In second integral do not take the terms containing x i.e. take only those terms of N which are free from x. If no such term is available then second integrals may not be considered.

(iii) c is arbitrary constant of Integration.

Solved Examples:-

1) Solve $\left(5x^4 + 6x^2y^2 - 8xy\right)dx + \left(4x^3y - 12x^2y^2 - 5y^4\right)\cdot dy = 0$

Soln: The given equation is:

$$\left(5x^4 + 6x^2y^2 - 8xy^3\right)dx + \left(4x^3y - 12x^2y^2 - 5y^4\right)dy = 0 -----(1)$$

$$\therefore M = 5x^4 + 6x^2y^2 - 8xy^3$$

$$N = 4x^3y - 12x^2y^2 - 5y^4$$

$$\therefore \frac{\partial M}{\partial Y} = \frac{\partial}{\partial Y}\left(5x^4 + 6x^2y^2 - 8xy^3\right)$$

$$= 0 + 12x^2y - 24xy^2$$

$$\therefore \frac{\partial M}{\partial Y} = 12x^2y - 24xy^2$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(4x^3y - 12x^2y^2 - 5y^4\right)$$

$$= 12x^2y - 24xy^2 - 0$$

$$\therefore \frac{\partial N}{\partial x} = 12x^2y - 24xy^2$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

$\therefore$ Its solution is given by

$$\int M(\text{treat y constant})dx + \int N(\text{terms free from x})\cdot dy = c$$

$$\therefore \int\left(5x^4 + 6x^2y^2 - 8xy^3\right)dx + \int\left(-5y^4\right)\cdot dy = c$$

$$\therefore 5\cdot\frac{x^5}{5} + 6y^2\cdot\frac{x^3}{3} - 8y^3\cdot\frac{x^2}{2} - 5\cdot\frac{y^5}{5} = c$$

$$x^5 + 2x^3y^2 - 4x^2y^3 - y^5 = c$$

Which is the required general solution

(2) Solve :

$$\frac{dy}{dx} = -\frac{4x^3y^2 + y\cos xy}{2x^4y + x\cos xy}$$

The given equation is

$$\frac{dy}{dx} = -\frac{4x^3y^3 + y\cos xy}{2x^4y + x\cos xy}$$

$$\therefore \left(4x^3y^2 + y\cos xy\right)dx + \left(2x^4 + y\cos xy\right)dy = 0..........(1)$$

Comparing with Mdx+Ndy=0; we have

$$M = 4x^3y^2 + y\cos xy$$

$$N = 2x^4y + x\cos xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(4x^3y^2 + y\cos xy\right)$$

$$\frac{\partial M}{\partial y} = 8x^3y^2 + \cos xy - xy\sin xy$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial}{x}\left(2x^4 y \ \ + y\cos xy\right)$$

$$\frac{\partial N}{\partial x} = 8x^3 y \ \ + \cos xy - xy\sin xy$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int \text{Min}(\text{treat y constant})dx + \int N(\text{terms free from x}) \cdot dy = c$$

$$\left(4x^3y^2 + y\cos xy\right)dx + \int o dy = c$$

$$4y^2 \int x^3 dx + y\int \cos xy = c$$

$$4y^2 \cdot \frac{x^4}{4} + y\frac{\sin xy}{y} = c$$

$$\therefore \ x^4y^2 + \sin xy = c$$

Which is the required general solution

(3) Solve: $\left(x - 2e^y\right)dy + \left(y + x\sin x\right)dx = 0$

The equation given is

$$\left(x - 2e^y\right)dy + \left(y + x\sin x\right)dx = 0$$

$$\therefore \ (y + x\sin x)dx + \left(x - 2e^y\right)dy = 0 - - - - - (1)$$

Comparing with Mdx+Ndy=0; we have

$$M = y + x\sin x$$

$$N = x - 2e^y$$

$$\therefore \ \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y + \sin x)$$

$$\therefore \ \frac{\partial M}{\partial y} = 1$$

$$\therefore \ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(x - 2e^y\right)$$

$$\therefore \ \frac{\partial N}{\partial x} = 1$$

$$\therefore \ \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence differential equation (1) is exact

Its solution is given by

$$\int M(\text{treat y constant})dx + \int N(\text{terms free from x}) \cdot dy = c$$

$$\therefore \ \int (y + x\sin x)dx + \int \left(-2 \cdot e^y\right) \cdot dy = c$$

$$\therefore xy + \left[x(-\cos x) + \sin x\right] - 2 \cdot e^y = c$$

Which is the required general solution

(4) Solve: $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + \left(x + \log x - x\sin y\right)dy = 0$

The given equation is

$$\left[y\left(1 + \frac{1}{x}\right) + \cos y\right]dx + \left(x + \log x - x\sin y\right) \cdot dy = 0 - - - - (1)$$

Comparing with Mdx+Ndy=0; we have

$$M = y\left(1 + \frac{1}{x}\right) + \cos y$$

$$N = x + \log x - x\sin y$$

$$\therefore \ \frac{\partial M}{\partial Y} = \frac{\partial}{\partial y}\left(y\left(1 + \frac{1}{x}\right) + \cos y\right)$$

$$\frac{\partial M}{\partial Y} = 1 + \frac{1}{x} - \sin y$$

$$\therefore \ \frac{\partial N}{\partial x} = \frac{\partial}{\partial x} \cdot \left(x + \log x - x\sin y\right)$$

$$\therefore \ \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial M}{\partial Y} = \frac{\partial N}{\partial x}$$

Hence the differential equation (1) is exact

Its solution is given by

$$\int M(\text{treat y constant})\,dx + \int N(\text{terms free from x})\,dy = c$$

$$\int\left(y\cdot\left(1+\frac{1}{x}\right)+\cos y\right)dx + \int o\,dy = c$$

$$y\cdot\int\left(1+\frac{1}{x}\right)dx + \int \cos y\,dy = c$$

$$y(x+\log x) + x\cos y = c$$

Which is the required general solution

(4) Solve: $$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

The given equation is

$$\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{(y\cos x + \sin y + y)}{(\sin x + x\cos y + x)}$$

$$\therefore (\sin x + x\cos y + x)dy = -(y\cos x + \sin y + y)dx$$

$$\therefore (y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)dy = 0 \text{ ----(1)}$$

Comparing with Mdx+Ndy=0; we have

$$M = y\cos x + \sin y + y$$

$$N = \sin x + x\cos y + x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(y\cos x + \sin y + y)$$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1$$

$$\therefore \frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(\sin x + x\cos y + x)$$

$$\therefore \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence the differential equation (1) is exact

Its solution is given by

$$\int M(\text{treat y constant})\,dx + \int N(\text{terms free from x})\,dy = c$$

$$\therefore \int [y\cos x + \sin y + y]\,dx + \int o\,dy = c$$

$$\therefore y\cdot\int y\cos x\cdot dx + \sin y\cdot\int dx + y\cdot\int dx = c$$

$$\therefore y\sin x + x\sin y + xy = c$$

Which is the require general solution

Check Point :

Solve:(1) $$\left(a^2 - 2xy - y^2\right)dx - (x+y)^2\cdot dy = 0$$

Ans. $$a^2x - x^2y - xy^2 - \frac{y^3}{3} = c$$

(2) $$\left(1+e^{x/y}\right)dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$$

Ans. $$x + y\cdot e^{x/y} = c$$

(3) $$[\cos x\cdot\tan y + \cos(x+y)]dx + [\sin x\cdot\sec^2 y + \cos(x+y)]dy = o$$

Ans. $$\sin x\cdot\tan y + \sin(x+y) = c$$

(4) $$\left(y^2e^{xy^2} + 4x^3\right)dx + \left(2xy\cdot e^{xy^2} - 3y^2\right)dy = 0$$

Ans. $$e^{xy^2} + x^4 - y^3 = c$$

(5) $$[1+\log(xy)]dx + \left\{1+\frac{x}{y}\right\}dy = 0$$

Ans. $$y + x\log(xy) = c$$

(6) $$\left(2xy + e^y\right)dx + \left(x^2 + xe^y\right)\cdot dy = 0$$

Ans. $$x^2y + xe^y = c$$

(7) $$\left[y\sin(xy) + xy^2\cos(xy)\right]dx + \left[x\sin(xy) + x^2y\cos(xy)\right]dy = 0$$

Hint:- $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \sin xy + xy\cos(xy) + 2xy\cos(xy) - x^2y^2\sin(xy)$$

General solution is given by

$$xy\sin(xy) = c$$

7

Equation Reducible to Exact Equations

In some cases equations which are not exact can be converted to exact differential equaion by multiplying by some suitable factor called as Integrating factor.

Definition :-

If IeMDX +IeNDY=0 is exact

then Ie is said to be an integrating factor of the equation Mdx+ Ndy = 0

Rules of finding Integrating factor :-

Rule (1)

If the equation Mdx+Ndy=0 is homogeneous then $\frac{1}{Mx+Ny}$ is integrating factor

Solved Example :

1) Solve $(x^2y-2xy^2)dx - (x^3-3x^2y)=0$

Soln : The given equation is

$$(x^2y-2xy^2)dx - (x^3-3x^2y)=0........(1)$$

Which is a homomgeneous equation.

Comparing with Mdx +Ndy=0 ; we have

$$M = x^2y-2xy^2$$
$$N = -(x^3-3x^2y)$$
$$\therefore \text{I.f.} = \frac{1}{Mx+Ny}$$
$$= \frac{1}{x^3y-2x^2y-x^3y+3x^2y^2}$$

$$\therefore \frac{(x^2y-2xy^2)}{x^2y^2}dx - \frac{(x^3-3x^2y)}{x^2y^2}dy = 0 \text{ is exact}$$

$$\text{i.e.} \left(\frac{1}{y}-\frac{2}{x}\right)dx - \left(\frac{x}{y^2}-\frac{3}{y}\right)dy = 0 \text{ is exact}$$

Its general solution is given by

$$\int M(\text{treat y const})dx + \int N(\text{terms free from x})dy = c$$

$$\int\left(\frac{1}{y}-\frac{2}{x}\right)dx + \int\frac{3}{y}dy = c$$

$$\therefore \frac{1}{y}\cdot\int dx - 2\int\frac{1}{x}dx + 3\cdot\int\frac{1}{y}dy = c$$

$$\therefore \frac{x}{2} - 2\log x + 3\log y = c$$

Which is required general solution

Check Point :

1) Solve

i $(3xy^2-y^3)dx + (xy^2-2x^2y)dy = 0$

Hint : $\text{I.F.} = \frac{1}{x^2y^2}$

General solution is given by

$$\frac{cy^2}{x^3} = c^{y/x}$$

ii $(x^2-3xy+2y^2)dx + x(3x-2y)dy = 0$

Hint : $\text{I.F.} = \frac{1}{x^3}$

General solution is given by

$$x^2\log x + 3xy = y^2 + cx^2$$

Rule (II) : If the equation Mdx+Ndy=0 can be written as

$y\, f_1(xy),\quad N = xf_2(xy)\cdot dy = 0$

i.e. $M = y\, f_1(xy),\quad N = x\, f_2(xy)$

then $\dfrac{1}{Mx - Ny}$ is an integration factor.

Note : - $f_1(x\,y), f_2(x\,y)$ are functions of xy.

Solved Examples :-

1) Solve : $\left(x^2y^2+2\right)ydx+\left(2-2x^2y^2\right)xdy=0$

Soln : The equation is given by

$$\left(x^2y^2+2\right)ydx+\left(2-2x^2y^2\right)xdy=0$$

Comparing with Mdx+Ndy=0; we have

$$\therefore M = \left(x^2y^2+2\right)y$$

$$N = \left(2-2x^2y^2\right)\cdot x$$

$$\text{I.f.} = \frac{1}{Mx-Ny}$$

$$\therefore \text{I.f.} = \frac{1}{xy\left(x^2y^2+2-2+2x^2y^2\right)}$$

$$\text{I.f.} = \frac{1}{3x^3y^3}$$

$$\therefore \frac{\left(x^2y^2+2\right)y}{3x^3y^3}dx + \frac{\left(2-2x^2y^2\right)\cdot x}{3x^3y^3}dy = 0$$

$$\text{i.e}\left(\frac{1}{3x}+\frac{2}{3}\cdot\frac{1}{x^3y^2}\right)dx+\left(\frac{2}{3x^3y^3}-\frac{2}{3y}\right)\cdot dy = 0$$

which is a exact equation

$\therefore$ Its General solution is given by

$$\int M\,(\text{treat y constant})\,dx + \int N\,(\text{terms free from x})\,dy = c$$

$$\therefore \int\left(\frac{1}{3x}+\frac{2}{3}\cdot\frac{1}{x^3y^2}\right)dx+\int -\frac{2}{3y}\cdot dy = c$$

$$\therefore \frac{1}{3}\int\frac{1}{x}dx+\frac{2}{3y^2}\cdot\int\frac{1}{x^3}dx-\frac{2}{3}\int\frac{1}{y}\cdot dy = c$$

$$\therefore \frac{1}{3}\log x-\frac{2}{6x^2y^2}-\frac{2}{3}\log y = c$$

$$\therefore \log x - \frac{1}{x^2y^2} - 2\log y = c_1 \text{ where } c_1 = 2c$$

Check point :-

1) solve :

(i) $\left(x^2y^2+xy+1\right)y\cdot dx+\left(x^2+y^2-xy+1\right)xdy=0$

Hint : I.F. $\dfrac{1}{2x^2y^2}$

G.S. is given by

$$xy+\log x-\frac{1}{xy}-\log y = c$$

2) $y\left(xy+2x^2y^2\right)+x\left(xy-x^2y^2\right)dy=0$

Ans: $x^2 = cy\cdot e^{\frac{1}{xy}}$

Rule (III) :

Let Mdx+ Ndy=0 be the given differential equation

If $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N} = a$ function of x alone. Say f(x) then is integrated. factor.

Solved Examples :-

1) Solve :

Soln : The given equation is

$$\left(x^4e^x-2mxy^2\right)dx+2mx^2y\,dy=0$$

Check Point :

1) solve :

1) $\left(x^2y^2 + x\right)dx + xy \cdot dy = c$

$\text{Hint If} = e^{\log x} = x$

General solution is given by

$4x^4 + 4x^3 + 6x^2y^2 = c$

ii) $\left(x^2 - 2x + 2y^2\right)dx + 2xy\,dy = c$

$\text{Hint : If} = x$

General solution is given by

$3x^4 - 8x^3 + 12x^3y^2 = c$

Rule (IV)

Let Mdx+Ndy=0 be the given diferential equation.

$$\text{If } \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \text{a function of } y \text{ alone} = f(y)$$

$$\therefore \text{I.F.} = e^{\int f(y)\,dy}$$

Solved examples :-

1) solve $\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$

Soln :- The given equation is

$$\left(y^4 + 2y\right)dx + \left(xy^3 + 2y^4 - 4x\right)dy = 0$$

comparing with Mdx+Ndy=0; we get

$$M = y^4 + 2y$$

$$N = xy^3 + 2y^4 - 4x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(y^4 + 2y\right)$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(xy^3 + 2y^4 - 4x\right)$$

$$\frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$$

$$= \frac{-3 \cdot (y^3 + 2)}{y(y^3 + 2)}$$

$$= -\frac{3}{y} = \text{function of } y \text{ alone}$$

$$\therefore \text{I.F.} = e^{\int f(y)dy}$$

$$= e^{-3 \cdot \int \frac{1}{y} dy}$$

$$= e^{-3 \log y}$$

$$= e^{\log\left(\frac{1}{y^3}\right)}$$

$$\text{I.F.} = \frac{1}{y^3}$$

$$\therefore \frac{(y^4 + 2y)}{y^3} dx + \frac{(xy^3 + 2y^4 - 4x)}{y^3} \cdot dy = 0$$

Which is exact differential equation

Comparing with Mdx+ Ndy=0; we get

$$M = y + \frac{2}{y^2}$$

$$N = x + 2y - 4\frac{x}{y^3}$$

General solution is given by

$$\int M(\text{treat } y \text{ constant})\, dx + \int N\,(\text{terms free from } x)\, dy = c$$

$$\therefore \int \left(y + \frac{2}{y^2} \right) dx + 2 \cdot \int y\, dy = c$$

$$\therefore \left(y + \frac{2}{y^2} \right) \int dx + 2\frac{y^2}{2} = c$$

$$\left(y + \frac{2}{y^2} \right) x + y^2 = c$$

Which is required general solution.

Check Point :-

Solve :

i) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$

Hint : $\text{I.F.} = \frac{1}{y^4}$ ....

General solution is given by

$$x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c$$

ii) $x^2y^3dx + (x^3y - 2)dy = 0$

Ans $3x^3y - 2y - 6 = cy \cdot e^{3/y}$

Linear Equation And Equations Reducible To Linear Form :

The first order and first degree linear - diferential equation is of the type

$$\frac{dy}{dx} + py = Q$$

Where y is dependent variable and x is independent variable. and p& Q are functions of x only. (may be constant)

The above diferential equation is known as Leibnitz's linear differential equation.

Working Rule :

1) Consider linear differential equation.

$$\frac{dy}{dx} + py = Q$$

Where P and are function of x or constants only

Its integrating factor is given by

$$I.F. = e^{\int p dx}$$

$$\frac{dx}{dy} + p_1 x = Q_1$$

Its solution is given by

$$y \cdot (I.F.) = \int Q \cdot (I.F.) dx + c$$

Where c is arbitrary constant.

2) For linear differential equation

Where p_1 and Q_1 are functions of y or constants only

Its integrating factor is given by

$$\therefore I.F. = e^{\int p_1 dy}$$

Its so;ution is given by

$$x \cdot (IF) = \int Q\,(IF)\, dy + c$$

Where c is arbitrary constant.

Solved Examples :-

1) Solve : $(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$

Soln : The given equation is

$$(x+1)\frac{dy}{dx} - y = e^x (x+1)^2$$

throughout by $(x+1)$ we have

$$\therefore \frac{dy}{dx} - \frac{1}{(x+1)} \cdot y = e^x (x+1) \text{........(1)}$$

This is of the type

$$\therefore \frac{dy}{dx} + py = Q$$

Hence equation (1) is linear differential equation.

Where $P = -\frac{1}{(x+1)}, Q = e^x (x+1)$

$$\therefore I.F. = e^{\int p dx}$$

$$= e^{-\int \frac{1}{x+1} dx}$$

$$= e^{-\log (x+1)}$$

$$I.F. = e^{\log\left(\frac{1}{x+1}\right)}$$

$$I.F. = \frac{1}{x+1}$$

Hence the solution of differential equation (1) is

$$y \cdot (I.F.) = \int Q\,(IF)\, dx + c$$

$$\therefore y \cdot \frac{1}{x+1} = \int e^x (x+1) \frac{1}{(x+1)} dx + c$$

$$\therefore \frac{y}{x+1} = \int e^x \cdot dx + c$$

$$\therefore \frac{y}{x+1} = e^x + c$$

$$\therefore y = (e^x + c) \cdot (x+1)$$

Which is the required solution.

2) Solve :

$$\left(1+y^2\right)dx=\left(\tan^{-1}-x\right)dy$$

Soln : The given equation is

$$\therefore\ \left(1+y^2\right)dx=\left(\tan y^{-1}-x\right)dy$$

$$\therefore \frac{dx}{dy}=\frac{\tan y^{-1}-x}{1+y^2}$$

$$\therefore \frac{dx}{dy}=\frac{\tan^{-1}y}{1+y^2}-\frac{1}{1+y^2}\cdot x$$

$$\therefore \frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{\tan^{-1}y}{1+y^2}\text{..........(1)}$$

This is of the type

$$\frac{dx}{dy}+px=\theta$$

$$\text{Where } p=\frac{1}{1+y^2},\theta=\frac{\tan^{-1}y}{1+y^2}$$

Hence equation (i) is a linear differential equation

$$\therefore I.f=e^{\int pdy}$$

$$=e^{\int\frac{1}{1+y^2}\cdot dy}$$

$$I.F.=e^{\tan^{-1}y}$$

The solution of differential equation (i) is

$$x(I.F.)=\int\theta\,(I.F.)dy+c$$

$$\therefore x\cdot e^{\tan\text{-}1y}=\int\frac{\tan^{-1}y}{1+y^2}\cdot e^{\tan^{-1}y}\cdot dy+c$$

consider the integral

$$\int\frac{\tan^{-1}y}{1+y^2}e^{\tan^{-1}y}\cdot dy$$

put $z=\tan^{-1}y$

Differentiating with respect to z

$$1=\frac{1}{1+Y^2}\cdot\frac{dy}{dz}$$

$$\therefore\frac{1}{1+Y^2}\cdot dy=dz$$

$$\therefore\int z\cdot e^z\cdot dz$$

$$=z\cdot\int e^z\cdot dz-\left(\int\frac{d}{dz}z\int e^z\cdot dz\right)dz$$

$$=z\cdot e^z-\int I\cdot e^z\cdot dz$$

$$=z\cdot e^z-e^z$$

$$=e^z(z-1)$$

put $z=\tan^{-1}y$

$$=e^{\tan^{-1}y}\left(\tan^{-1}y-1\right)$$

$\therefore$ solution is given by

$$x\cdot e^{\tan^{-1}y}=e^{\tan^{-1}y}\left(\tan^{-1}y-1\right)+c$$

$$\therefore x=\tan^{-1}y-1+c\cdot e^{-\tan^{-1}y}$$

Which is the required solution.

3) Solve : $x\left(1-x^2\right)\frac{dy}{dx}+\left(2x^2-1\right)y=x^3$

Soln: The given equation is

$$x\left(1-x^2\right)\frac{dy}{dx}+\left(2x^2-1\right)y=x^3$$

$\div$ through out by $x(1-x^2)$ we have

$$\therefore\frac{dy}{dx}+\frac{\left(2x^2-1\right)}{x\left(1-x^2\right)}y=\frac{x^3}{x\left(1-x^2\right)}\text{..........(1)}$$

Hence equation (1) is linear in depedent variable y

This is of the type

$$\frac{dy}{dx}+py=Q$$

$$\text{where } P=\frac{(2x^2-1)}{x(1-x^2)},Q=\frac{x^3}{x(1-x^2)}$$

$$\therefore \text{I.F.}=e^{\int pdx}$$

$$\text{Let } P=\frac{2x^2-1}{x(1-x)(1+x)}$$

$$P=-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}$$

(By partial fraction)

$$\therefore \text{IF}=e^{\int\left[-\frac{1}{x}+\frac{1}{2(1-x)}-\frac{1}{2(1+x)}\right]\cdot dx}$$

$$=e^{-\log x+\frac{1}{2}\log(1-x)-\frac{1}{2}\log(1+x)}$$

$$=e^{-\left[\log x\cdot\sqrt{1-x^2}\right]}$$

$$=e^{\log\cdot\left[x\cdot\sqrt{1-x^2}\right]^{-1}}$$

$$\text{IF}=\frac{1}{x\sqrt{1-x^2}}$$

Hence solution of differential equation (i) is

$$y(\text{IF})=\int Q(\text{IF})dx+c$$

$$\therefore y\cdot\frac{1}{x\sqrt{1-x^2}}=\int\frac{x^2}{(1-x^2)}\cdot\frac{1}{x\sqrt{1-x^2}}\cdot dx+c$$

$$=\int\frac{x}{(1-x^2)^{3/2}}\cdot dx+c$$

$$=-\frac{1}{2}\cdot\int(-2x)(1-x^2)^{3/2}\cdot dx+c$$

$$=-\frac{1}{2}\left[\frac{(1-x^2)^{-1/2}}{-1/2}\right]+c$$

$$\left\{\int f^n\cdot f^1=\frac{f^{n+1}}{n+1}\right.$$

$$\therefore \frac{y}{x\sqrt{1-x^2}}=\frac{1}{\sqrt{1-x^2}}+c$$

$$\therefore y=x+cx\sqrt{1-x^2}$$

Which is the required solution.

4) Solve :

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy}=1$$

Soln : The given equation is

$$\left[\frac{e^{-2\sqrt{x}}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right]\cdot\frac{dx}{dy}=1$$

$$\therefore \frac{dy}{dx}+\frac{y}{\sqrt{x}}=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}\text{........(1)}$$

Which is of the type

$$\frac{dy}{dx}+py=Q$$

$$\text{where } P=\frac{1}{\sqrt{x}},Q=\frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

The equation (1) is linear in y

$$\therefore \text{I.F.}=e^{\int pdx}$$

$$=e^{\int\frac{1}{\sqrt{x}}\cdot dx}$$

$$\text{I.F.}=e^{2\sqrt{x}}$$

Hence the solution of differential equation (1) is

$$y \cdot (IF) = \int \phi \cdot (IF) dx + c$$

$$y \cdot e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \cdot e^{2\sqrt{x}} \cdot dx + c$$

$$= \int \frac{1}{\sqrt{x}} dx + c$$

$$y \cdot e^{2\sqrt{x}} = 2\sqrt{x} + c$$

Which is the required general solution.

5) Solve :

$$(1+y^2) + (x - e^{\tan^{-1} y}) \cdot \frac{dy}{dx}$$

Soln : The given equation is

$$(1+y^2) + (x - e^{\tan^{-1} y}) \cdot \frac{dy}{dx}$$

$$\therefore (x - e^{\tan^{-1} y}) \cdot \frac{dy}{dx} = -(1+y^2)$$

$$\therefore x - e^{\tan^{-1} y} = -(1+y^2) \cdot \frac{dx}{dy}$$

$$\therefore \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1} y}}{1+y^2} \text{........(1)}$$

Which is of the type

$$\frac{dx}{dy} + px = \phi$$

$$\text{where } p = \frac{1}{1+y^2}, \phi = \frac{e^{\tan^{-1} y}}{1+y^2}$$

The equation (1) is linear differential equation

Hence

$$IF = e^{\int p dy}$$

$$= e^{\int \frac{1}{1+y^2} \cdot dy}$$

$$IF = e^{\tan^{-1} y}$$

Hence solution of differential equation (1) is given by

$$x \cdot (IF) = \int Q\ (IF)\ dy + c$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1+y^2} \cdot e^{\tan^{-1} y} \cdot dy + c$$

$$x \cdot e^{\tan^{-1} y} = \int \frac{e^{2\tan^{-1} y}}{1+y^2} \cdot dy + c \text{.............(2)}$$

put $\tan^{-1} y = t$

$$\therefore \frac{1}{1+y^2} \cdot dy = dt$$

$\therefore$ equation (2) becomes

$$x \cdot e^{\tan^{-1} y} = \int e^{2t} \cdot dt + c$$

$$x \cdot e^{\tan^{-1} y} = \frac{e^{2t}}{2} + c$$

put $t = \tan^{-1} y$

$$\therefore x \cdot e^{\tan^{-1} y} = \frac{e^{2\tan^{-1} y}}{2} + c$$

$$\therefore 2x \cdot e^{\tan^{-1} y} = e^{2\tan^{-1} y} + c_1 \text{ where } c_1 = 2c$$

Which is the required general solution

Check Points :

1) Solve

i) $(2y + x^2) dx = x dy$

Ans : $y = x^2 \log(cx)$

ii) $\frac{dy}{dx} + \frac{y}{1-x} = x^2 - x$

Ans : $2y = (1-x)(c^2 - x^2)$

iii) $(x^2+1) \cdot \frac{dy}{dx} = x^3 - 2xy + x$

Ans : $(x^2+1)y = \frac{x^4}{4} + \frac{x^2}{2} + c$

iv) $\frac{dy}{dx}+\frac{x}{(1-x^2)^{3/2}}\cdot y=\frac{x\left(1+\sqrt{1-x^2}\right)}{(1-x^2)^2}$

Hint I.F. $= e^{-\frac{1}{\sqrt{1-x^2}}}$

$y=\frac{1}{\sqrt{1-x^2}}+c\cdot c^{\frac{1}{\sqrt{1-x^2}}}$

v) $dx+xdy=e^{-y}\sec^2\cdot dy$

Hint : I.F. $= e^y$

$x\cdot e^y=\tan y+c$

vi) $x\cos x\cdot\frac{dy}{dx}+(\cos x-x\sin x)\cdot y=1$

Hint : I.F. $=\frac{x}{\sec x}$

$xy\cos x=x+c$

vii) $(x^2+1)^3\cdot\frac{dy}{dx}+4x\cdot(x^2+1)^2\cdot y=1$

Hint : If $=(x^2+1)^2$

$(x^2+1)^2\cdot y=\tan^{-1}x+c$

viii) $(x+y+1)\cdot\frac{dy}{dx}=1$

Hint : I.F. $= e^{-y}$

$x+y+2=c\cdot e^y$

ix) $(x+2y^3)\cdot dy=ydx$

Hint I.F. $=\frac{1}{y}$

$x=y^3+cy$

Equations reducible to linear form :

I) Bernoullis Equation :

The equation of the form

$$\frac{dy}{dx}+py=Q\cdot y^n$$

is called as Bernoullis Equations.

$$\frac{dy}{dx}+py=Q\cdot y^n$$

is called as Bernoulli's equations

$\div$ throughout by y^n, we get

$$\therefore y^{-n}\cdot\frac{dy}{dx}+P\cdot y^{1-n}=Q.........(1)$$

Let $y^{1-n}=u$

$$\therefore (1-n)\cdot y^{-n}\cdot\frac{dy}{dx}=\frac{du}{dx}$$

using equation (1) we get

$$\therefore \frac{1}{1-n}\cdot\frac{dy}{dx}+Pu=Q$$

$$\therefore \frac{du}{dx}+(1-n)\cdot pu=(1-n)Q$$

$$\frac{dx}{dy}+Px=Q\cdot x^n$$

Which is Bernoulli's differential equation and can be solved.

Note : The equation is also Bernoulli's equation

we divide by x^n and substitute u= x^{1-n} and proceed.

Solved Examples :-

1) Solve : $\frac{dy}{dx}+\frac{y}{x}=xe^x\cdot y^2$

$$\frac{dy}{dx}+\frac{y}{x}=xe^{x}\cdot y^{2}.............(1)$$

Which is of the type

$$\frac{dy}{dx}+Py=Q\cdot y^{n}........$$

Where $p=\frac{1}{x}, Q=xe^{x}, n=2$

Equation (1) is Bernoulli's differential equation

$\div$ throughtout by y^2, we get

$$\therefore y^{-2}\cdot\frac{dy}{dx}+\frac{1}{x}\cdot y^{-1}=x\cdot e^{x}.................(2)$$

Put $y^{-1}=u$

Differentiating with respect to x

$$\therefore -1\cdot y^{-2}\cdot\frac{dy}{dx}=\frac{du}{dx}$$

$$\therefore y^{-2}\cdot\frac{dy}{dx}=-\frac{du}{dx}$$

using equation 2 we get

$$\therefore -\frac{du}{dx}+\frac{1}{x}\cdot u=x\cdot e^{x}$$

$$\therefore \frac{du}{dx}-\frac{1}{x}\cdot u=x\cdot e^{x}$$

Which is linear differential equation.

where $p=\frac{1}{x}, Q=-x\cdot e^{x}$

$$\therefore \text{I.F.}=e^{\int Pdx}$$

$$=e^{-\int\frac{1}{x}dx}$$

$$=e^{-\log x}$$

$$\text{I.F.}=e^{\log\left(\frac{1}{x}\right)}$$

$$\therefore \text{I.F.}=\frac{1}{x}$$

Hence,General solution is given by

$$u\cdot(IF)=\int Q\,(IF)\,dx+c$$

$$\therefore u\cdot\frac{1}{x}=\int -x\cdot e^{x}\,\frac{1}{x}\,dx+c$$

Put $u=y^{-1}$

$$\therefore y^{-1}\cdot\frac{1}{x}=-\int e^{x}dx+c$$

$$\frac{1}{xy}=-e^{x}+c$$

Which is the required solution.

2) Solve :

$$xy\left(1+xy^{2}\right)\cdot\frac{dy}{dx}=1$$

Soln : The given equation is

$$xy\cdot\left(1+xy^{2}\right)\cdot\frac{dy}{dx}=1$$

$$\therefore \frac{dx}{dy}=xy+x^{2}y^{3}$$

$$\therefore \frac{dx}{dy}-xy=x^{2}y^{3}.........(1)$$

which is of the type,

$$\frac{dx}{dy}+px=Q\cdot x^{n}$$

where $p=-y, Q=y^{3}, n=2$

Equation 1 is a Bernoulli's differential equation

$\div$ through out by x^2, we get

$$\therefore x^{-2}\cdot\frac{dx}{dy}-x^{-1}\cdot y=y^{3}.........(2)$$

Let $x^{-1}=u$

$$\therefore -x^{-2}\cdot\frac{dx}{1y}=\frac{du}{dy}$$

$$\therefore x^{2}\cdot\frac{dx}{dy}=-\frac{du}{dy}$$

equation (2) becomes

$$-\frac{du}{dy}-uy=y^{3}$$

$$\therefore \frac{du}{dy}+uy=-y^{3}$$

which is a linear differential equation.

$$p=y,\quad \phi=-y^{3}$$

$$\therefore I.F.=e^{\int pdy}$$

$$=e^{\int y\cdot dy}$$

$$I.F.=e^{y^2/2}$$

Hence general solution is given by

$$u\cdot(IF)=\int Q\cdot(IF)\cdot dy+c$$

$$u\cdot e^{y^2/2}=\int -y^{3}\cdot e^{y^2/2}\cdot dy+c$$

$$\text{Let } \frac{y^2}{2}=t$$

$$\therefore y\,dy=dt$$

$$\therefore u\cdot e^{y^2/2}=-\int 2t\cdot e^{t}\cdot dt+c$$

$$u\cdot e^{y^2/2}=-2\left[t\cdot e-e^{t}\right]+c$$

$$\text{Put } u=x^{-1},\ t=y^2/2$$

$$\therefore \frac{1}{x}\cdot e^{y^2/2}=-2\left[\frac{y^2}{2}\cdot e^{y^2/2}-e^{y^2/2}\right]+c$$

$$\therefore \frac{1}{x}\cdot e^{y^2/2}=-y^{2}\cdot e^{y^2/2}+2\cdot e^{y^2/2}+c$$

$$\therefore \frac{1}{x}\cdot e^{y^2/2}+y^{2}\cdot e^{y^2/2}-2\cdot e^{y^2/2}=c$$

Which is the required general solution.

Check points:-

i) solve:-

i) $\frac{dy}{dx}-y\tan x=y^{4}\sec x$

Hint : $If=\sec^{3}x$

$\frac{\sec^{3}x}{y^{3}}+3\tan x+\tan^{3}x=c$

ii) $\frac{dy}{dx}-xy=y^{2}\cdot e^{-x^2/2}\cdot\log x$

Hint : $I.F.=e^{x^2/2}$

$\frac{1}{y}\cdot e^{-x^2/2}+x\log x-x=c$

iii) $xy-\frac{dy}{dx}=y^{3}\cdot e^{-x^{2}}$

Hint : $If=e^{x^{2}}$

$e^{x^{2}}=y^{2}(2x+c)$

iv) $x\frac{dy}{dx}+3y=x^{4}e^{1/x^2}\cdot y^{3}$

Ans: $y^{2}+x^{6}\left(e^{1/x^2}+c\right)=1$

iv) $2xdx-y^{2}(y^{3}+x^{2})\cdot dy=0$

Hint : $If=e^{-y^3/3}$

$x^{2}=c\cdot e^{y^3/3}-y^{3}-3$

vi) $$\frac{dy}{dx} = e^{x-y}\left(e^x - e^y\right)$$

$$\text{Hint}: IF = e^{e^x}$$

$$e^y = c \cdot e^{-e^x} + e^x - 1$$

$$\frac{dy}{dx} = 2y(1-2xy)$$

$$\text{Hint}: \text{-I.F.} = e^{2x}$$

$$\frac{1}{y} = (2x-1) + c \cdot e^{-2x}$$

(II) Equation of the type :

The equation $$f^1(x)\cdot\frac{dy}{dx} + p\cdot f(y) = Q$$

Where P and Q are functions of x can be reduced to linear by substituting f(y) =u and equation becomes

$$\frac{du}{dx} + pu = Q$$

Similarly the equation

$$f^1(x)\cdot\frac{dy}{dx} + p\, f(x) = Q$$

Can be reduced to linear by substituting f (x)=u

Solved Examples :-

1) Solve : $$\sin y\cdot\frac{dy}{dx} = (1 - x\cos y)\cdot\cos y$$

Soln :

The given equation is $$\sin y\cdot\frac{dy}{dx} = (1 - x\cos y)\cdot\cos y$$

$$\therefore \sin y\cdot\frac{dy}{dx} = \cos y - x\cos^2 y$$

$\div$ throughout by $\cos^2 y$, we get

$$\therefore \frac{\sin y}{\cos^2 y}\cdot\frac{dy}{dx} = \frac{\cos y}{\cos^2 y} - x$$

$$\therefore \sec y\cdot\tan y\cdot\frac{dy}{dx} - \sec y = -x\text{.........(1)}$$

which is of the form

$$f^1(y)\cdot\frac{dy}{dx} + pf(y) = Q$$

where $f(y) = \sec y$, $p = -1$. $Q = -x$

Let $\sec y = u$

Differentiating with respect to x

$$\therefore \sec y\cdot\tan y\cdot\frac{dy}{dx} = \frac{du}{dx}$$

$\therefore$ equation (1) becomes

$$\therefore \frac{du}{dx} - u = -x$$

Which is a linear differential equation.

Where $p = -1, Q = -x$

$$\therefore \text{I.F.} = e^{\int p\,dx}$$

$$= e^{-dx}$$

$$\text{If.} = e^{-x}$$

Hence General solution is given by

$$u\cdot(IF) = \int Q\cdot(IF)\,dx + c$$

$$u\cdot e^{-x} = \int -x\cdot e^{-x}\cdot dx + c$$

$$= -\int x\cdot e^{-x}\cdot dx + c$$

$$u\cdot e^{-x} = -\left[x\left(-e^{-x}\right) - 1\cdot e^{-x}\right] + c$$

Put $u = \sec y$

$$\therefore \sec y = x + 1 + c\cdot e^x$$

Which is required general solution

Check Points :

1) Solve :

i) $\frac{dy}{dx}+\frac{1}{x}\tan y=\frac{1}{x^2}\tan y\cdot\sin y$

$\text{Hint} \div \text{throughout by } \tan y\cdot\sin y$

$\frac{1}{x\sin y}=\frac{1}{2x^2}+c$

ii) $\frac{dy}{dx}-x^3\cos^2 y=-x\sin 2y$

$\text{Hint}\quad \frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

$\div \text{ through out by } \cos^2 y$

$IF=e^{x^2}$

$2\tan y=x^2-1+c_1\cdot e^{-x^2}$

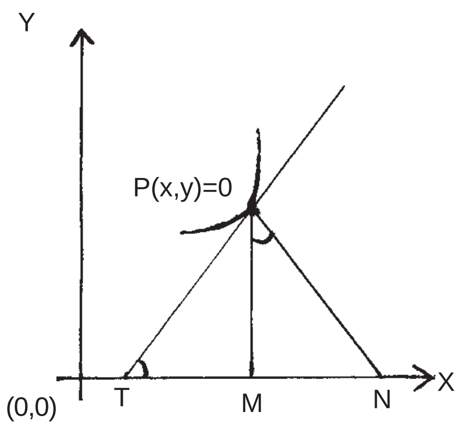
8

APPLICATIONS OF DIFFERENTIAL EQUATIONS

COMETRICAL APPLICATIONS :

tesian Co-ordinates :

Let $f\,(x_1 y) = 0$ be the equation of the curve Let p $(x_1 y_1)$ ie any point on it.



the tangent and normal at p meet X – axis in T and respectively.

Lef Pm $\perp$ X – axis

Let $\angle$ MTP = Ψ

$\therefore$ $\angle$ MTP = Ψ ...... [Geometrical Contruction]

Then,

Slope of Tangent at p = tan Ψ = $\left(\frac{dy}{dx}\right)(x_1, y_1)$

Equation of tangent at p is

$$y - y_1 = \left(\frac{dy}{dx}\right)(x - x_1)$$

X– intersept of tangent = $x_1 - y_1\left(\frac{dx}{dy}\right)p$

$$= x_1 - \frac{y_1}{\left(\frac{dy}{dx}\right)_p}$$

y –intersept of tangent = $y_1 - x_1\left(\frac{dy}{dx}\right)P$

Equation of the naormal at P is given by

$$y - y_1 = -\left(\frac{dx}{dy}\right)(x - x_1)$$

6) Lenght of tangent = PT = $y_1\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

7) Length of Normal at P = PN = $y_1\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

8) Length of Sub tangent = $\frac{y_1}{\left(\frac{dy}{dx}\right)}$

9) Length of Sub normal = $y_1 \cdot \left(\frac{dy}{dx}\right)$

10) If e is a radius of curvature at p then

$$e = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

SOLVED EXAMPLES :

1) Find the curve which passes through the points [2, 1] and [8, 2] for which sub tangent at any point varies as the abscissa of that point.

Soln : Let p [$x_1 y$] be a point on the curve

We know,

$$\text{subtangent} = \frac{y}{\frac{dy}{dx}}$$

From given condition -

$$\frac{y}{\frac{dy}{dx}} \propto x$$

$$\therefore \frac{y}{\frac{dy}{dx}} = kx.....[k = \text{constant}]$$

$$\therefore y = kx\frac{dy}{dx}$$

$$\frac{1}{x}dx = \frac{k}{y}dy$$

$$k = \frac{dy}{y} = \frac{1}{x}dx$$

which is in variable seperate form

integrate both side

$$\therefore k.\int \frac{1}{y}dy + \text{constant}$$

k. log y = log x + log c

$$\therefore \log\ y^k = \log(cx)$$

$$\therefore y^k = (cx).......(1)$$

The Curve passes through the points [2,1] and [6, 2]

put x = 2 , y = 1, in eq^n [1]

$$\therefore 1^k = cx2$$

$$\therefore 1 = cx2$$

$$\therefore c = \frac{1}{2}$$

put x = 8, y = 2, in eq^n [1]

$$\therefore 2^k = c \times 8$$

$$2^k = \frac{1}{2} \times 8^4$$

$$2^k = 4$$

$$\therefore 2^k = 2^2$$

$$\therefore K = 2$$

put Value of C and K in eq^n [1]

$$\therefore\ y^2 = \frac{1}{2}x$$

$$\therefore\ 2y^2 = x$$

which is the equation of the Curve.

2) Find the curves in which the length of the radius of curvature at any point is equal to two times the length of the normal at that point.

Solu. : Let p [x , y] be a point on the curve

We Known that,

$$\text{Radius of curvature} = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

Lenght of normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$\therefore$ From given condition -

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = 25\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\therefore \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} \bullet \left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = 25\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2}$$

$$\therefore 1+\left(\frac{dy}{dx}\right)^2 = 25 \bullet \frac{d^2y}{dx^2} \longrightarrow [1]$$

Let $\frac{dy}{dx} = z$

$$\therefore \frac{dy^2}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}(z)$$

$$= \frac{dz}{dy} \bullet \frac{dy}{dx}$$

$$= \frac{dz}{dy} \bullet z$$

$$\therefore \frac{d^2y}{dx^2} = z\frac{dz}{dy}$$

From eq^n (1)

$$\therefore\ 1 + 2^2 = 25 \cdot z \cdot \frac{dz}{dy}$$

$$\therefore\ 2zy \cdot \frac{dz}{dy} = 1 + 2^2$$

$$\therefore\ \frac{2z}{1 + 2^2} \cdot dz = \frac{1}{y} \cdot dy$$

which is in variable swparable form

$\therefore$ Integrate both side

$$\therefore\ \int \frac{2z}{1 + z^2}\, dz = \int \frac{1}{y} \cdot dy \text{ constant}$$

$$\therefore\ \log\left(1+z^2\right) = \log y + \log c$$

$$\therefore\ \log\left(1+z^2\right) = \log(cy)$$

$$1+z^2 = cy$$

$$\therefore\ z^2 = cy - 1$$

$$\therefore\ z = \sqrt{cy - 1}$$

Again put $z = \frac{dy}{dx}$

$$\therefore\ \frac{dy}{dx} = \sqrt{cy - 1}$$

$$\therefore\ \frac{1}{\sqrt{cy - 1}}\, dy = dx$$

Which is in variable seperable form

Where $c_2 = cc_1$

which is the eq^n of the curve.

Check Point :

1) Determine the curves for which sub normal is the arithmatic mean between the abscissa and the ordinate

[Hint : $y\dfrac{dy}{dz} = \dfrac{x}{z} + \dfrac{y}{z}$; simplify

Equation is homegeneous.

Ans : $(x + 2y) \cdot (x + y)^2 = c$

Physical Application :

Rectilinear Motion :

It is a Motion of a body of Mass in start moving from a fixed point O along a straight line OX under the action of a force F. Let p be the position of the body at any instant

where OP = X , then

1) velocity $v = \dfrac{dx}{dt}$

2) The acceleration $= \dfrac{dv}{dt}$

$$= \frac{d^2x}{dt^2}$$

$$= v \cdot \frac{dv}{dx}$$

By chain rule -

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{dv}{dx} \cdot v$$

$$= v \cdot \frac{dv}{dx}$$

3) Newton's second law of motion is given by

$f = ma$

$= m \cdot \dfrac{dv}{dt}$

$= m \cdot \dfrac{d^2x}{dt^2}$

$f = mv \cdot \dfrac{dv}{dx}$

where f = effective force

D' Alembert's principle :-

Algebric sum of the forces acting on a body along the given direction is equal to the product of mass and acceleration in that direction.

Solved examples :

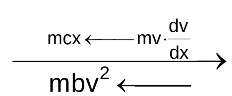
1) A moving body is opposed by a force per unit mass of a value CX and resistant per unit mass value bv, where X and V are the displacement and velocity of the particle at that instant. Show that the velocity of the particle. If it starts from rest, is given by.

$$v^2 = \frac{c}{2b^2}\left(1 - e^{2bx}\right) - \frac{cx}{b}$$

Soln. : Consider the motion

Step 1) :

Let m be the mass of the particle moving to right. Now the opposing forces mcx and mbv^2 will act to the left.



ie $-mcx$ and $-mbv^2$ are forces to the right

By D' Alembert's principle :- $mv\cdot\frac{dv}{dx}=-mcx-mbv^2$

step[2]

$$\therefore v\frac{dv}{dx}+bv^2=-cx\longrightarrow[1]$$

Let $v^2=z$

$$\therefore 2v\cdot\frac{dv}{dx}=\frac{dz}{dx}$$

$\therefore$ eq^n [1] becomes

$$\frac{1}{2}\cdot\frac{dz}{dx}+bz=-cx$$

$$\therefore \frac{dz}{dx}+2bz=-2cx$$

which is a linear equation in z

$\therefore$ p = 2b. Q = - 2cx.

$$\therefore I.F.=e^{\int pdx}$$

$$=e^{\int 2bdx}$$

$$I.F.=e^{2bx}$$

Its general solution is given by

$$z[IF]=\int Q\cdot(IF)dx+\text{constant}$$

$$\therefore ze^{2bx}=\int(-2cx)e^{2bx}\cdot dx+c_1$$

$$=-2c\cdot\int x\cdot e^{2bx}\cdot dx+c_1$$

$$=-2c\cdot\left[x\int e^{2bx}\cdot\left(dx-\int\frac{d}{dn}x\int e^{2bx}\cdot dx\right)\right]+c_1$$

$$=-2c\cdot\left[x\cdot\frac{e^{2bx}}{2b}\int 1\cdot\frac{e^{2bx}}{2b}dx\right]+c_1$$

$$ze^{2bx}=-2c\left[\frac{x\cdot e^{2bx}}{2b}-\frac{1}{2b}\cdot\frac{e^{2bx}}{2b}\right]+c_1$$

$$v^2\cdot e^{2bx}=\frac{-cx\cdot e^{2bx}}{b}+\frac{c}{2b^2}\cdot e^{2bx}+c_1$$

$$\therefore v^2=-\frac{cx}{b}+\frac{c}{2b^2}+c_1\cdot e^{-2bx}\longrightarrow[3]$$

[III] to find c_1 , we impose initial conditions

ie for x = o , v = o in eq^n [3]

$$o=0+\frac{c}{2b^2}+c_1$$

$$\therefore c_1=-\frac{c}{2b^2}$$

put values of c_1 in eq^n [3]

$$v^2=-\frac{cx}{b}+\frac{c}{2b^2}-\frac{c}{2b^2}\cdot e^{-2bx}$$

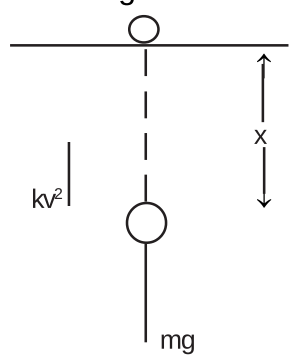
$$\therefore v^2=-\frac{c}{2b^2}\left(1-e^{-2bx}\right)-\frac{cx}{b}$$

2) A body of mass m. Falling from rest, is subject to the force of gravity and an air resistance proportional to the square of the velocity [ie kv^2]. If it falls through a distance x and possesses a velocity v at that instant show that

$$\frac{2k\,x}{m}=\log\left(\frac{a^2}{a^2-v^2}\right),\text{ where } mg=ka^2$$

soln :

Step □:1



Let the body of mass m fall from 'O'

The forces acting on the body are

1) Its weight mg acting vertically downwards.

2) The resistance kv^2 of the air acting vertically upwards.

The net forces acting on the body vertically downwards

$= mg - kv^2$ . . . . [$mg - ka^2$ given]

$= ka^2 - kv^2$

$= k [a^2 - v^2]$ . . . . . .[1]

Step [2] By D'Alembert's Priciple

$$mv \cdot \frac{dv}{dx} = k\left(a^2 - v^2\right)$$

$$\therefore \frac{v}{a^2 - v^2} \cdot dv = \frac{k}{m} \cdot dx$$

Which is in variable seperable form

integrating both sides

$$\therefore -\frac{1}{2} \cdot \int \frac{-2v}{a^2 - v^2} \cdot dv = \frac{k}{m} \cdot \int dx + c_1 \;. \;. \;. \left[c_1 = \text{ constant}\right]$$

$$\therefore -\frac{1}{2} \log\left(a^2 - v^2\right) = \frac{k}{m} x + c_1 \longrightarrow (2)$$

Step [3] To Final c_1 , we put initial conditions

ie when x = o , v = o.

$\therefore$ From (2)

$$\therefore -\frac{1}{2} \log a^2 = c_1$$

put value of c_1 in eq^n [2]

$$\therefore -\frac{1}{2} \log\left(a^2 - v^2\right) = \frac{k}{m} x - \frac{1}{2} \log a^2$$

$$\therefore -\frac{1}{2} \log\left(a^2 - v^2\right) = \frac{k}{m} x - \frac{1}{2} \log a^2$$

$$- \log\left(a^2 - v^2\right) = \frac{2kx}{m} - \log a^2$$

$$\therefore \log a^2 - \log\left(a^2 - v^2\right) = \frac{2kx}{m}$$

$$\therefore \frac{2kx}{m} = \log\left(\frac{a^2}{a^2 - v^2}\right)$$

Check Point :

1) A particle of Unit mass is projected upward with velocity u and the resistance of air produces a – retardation kv^2 and v is the velocity at any instant show that the velocity v with which the particle will return to the point of projection is given by

$$\frac{1}{v^2} = \frac{1}{u^2} + \frac{k}{g}$$

2) Determine the least velocity with which a particle must be projected vertically upwards so that it doesnot return to the Earth. Assume that it is acted upon by the gravitational attraction of the earth only.

Ans : Least Velocity $vo = \sqrt{2gR}$

R = Radius of earth

3) A paratroper and his parachute weigh 50 kg. At the instant parachute opens. He is Travelling vertically downward at the speed of 20 m/s. If the Air resistance varies directly as the instantenous velocity and its 20 Newtons. When the velocity is 10 m/s Find the limiting velocity, the position and the velocity of the paratrooper at any time "t".

$$v = 5\left[s - e^{-gt/25}\right] . . . = 25 \text{ m/s}$$

$$x = 5\left[st + \frac{25}{g} \cdot e^{-gt/25}\right] + c_1$$

$$x = 25t - \frac{125}{g}\left[1 - e^{-gt/25}\right]$$

Simple Electric Circuits :

The following Notations are frequently used. Units are given in Brackets .

$t\ (\text{seconds}) \longrightarrow$ Time

$q\ (\text{coulombs}) \longrightarrow$ Charge on capacitor

$i\ (\text{ampere}) \longrightarrow$ Current

$e\ (\text{volts}) \longrightarrow$ voltage

$R\ (\text{ohms}) \longrightarrow$ Resistance

$L\ (\text{Hentries}) \longrightarrow$ Induatance

$C\ (\text{Farads}) \longrightarrow$ capacitance

$\therefore$ Current is the rate of electricity

$$\therefore\ i = \frac{dq}{dt}$$

[II] Current at each point of a network is got from kirchoff's laws :

1) The algebric sum of the currents into any point is zero.

2) Around any closed path the algebric sum of the voltage drops in any speccific direction is zero.

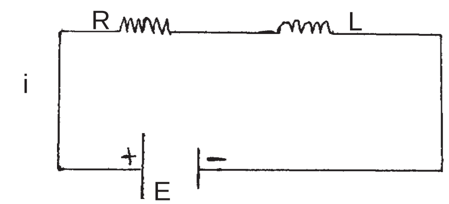
3) Voltage drops as current i flows through a resistance R is Ri ; through an induction L is $L\frac{di}{dt}$ and through a capacitor C is $\frac{q}{c}$.

Solved examples :

1) A constant emf E volts is applied to a ckt. containing a constant resistance. R ohms in series and a constant inductance L henries. It the initial current is zero , show that the current builds upto half its theoretical maximum in $\frac{L\log 2}{R}$ seconds.

Soln.

Step (1)



Let i be the current in the circuit at any time 't' .

The by Kirchoff's law, we have

$$E = L\cdot\frac{di}{dt} + Ri$$

$$\therefore\ L\frac{di}{dt} + Ri = E$$

$$\therefore\ \frac{di}{dt} + \frac{R}{L}\cdot i = \frac{E}{L} \longrightarrow (1)$$

Which is a linear equation in i .

$$\therefore\ P = \frac{R}{L}\ \cdot\ \ Q = \frac{E}{L}$$

$$\therefore\ \text{I.F.} = e^{\int p\,dt}$$

$$= e^{\int \frac{R}{L}\cdot dt}$$

$$\text{I.F.} = e^{\frac{R}{L}\cdot t}$$

$\therefore$ The general solution is given by

$$i.(IF) = \int Q.(IF)\cdot dt + \text{constant}$$

$$i.\,e^{\frac{R}{L}\cdot t} = \int \frac{E}{L}\cdot e^{\frac{R}{L}\cdot t}\cdot dt + c$$

$$= \frac{E}{\not L}\cdot\frac{\not L}{R}\cdot e^{\frac{R}{L}\cdot t} + c$$

$$i.\,e^{\frac{R}{L}\cdot t} = \frac{E}{R}\cdot e^{\frac{R}{L}\cdot t} + c$$

$$\therefore i = \frac{E}{R} + c\cdot e^{\frac{-R}{L}\cdot t} \longrightarrow (2)$$

To find c , we impose initial conditions

i.e. at $t = o , i = o$

$$\therefore O = \frac{E}{R} + C$$

$$\therefore C = -\frac{E}{R}$$

$\therefore$ Equation (2) becomes

$$i = \frac{E}{R} - \frac{E}{R} \cdot e^{-\frac{R}{L} \cdot t}$$

$$\therefore i = \frac{E}{R}\left(1 - e^{-\frac{R}{L} \cdot t}\right) \longrightarrow (3)$$

Which is the expression for i at any time t.

Now as t increases $e^{-\frac{R}{L} \cdot t}$ decreases i increases and its maximum value is $\frac{E}{R}$

Step (2)

Let the current in the circuit be half its theoretical maximum after a time T seconds then.

From eqn (3) $$\frac{1}{2}\frac{E}{R} = \frac{E}{R}\left(1 - e^{-\frac{R}{L} \cdot t}\right)$$

$$\therefore \frac{1}{2} = 1 - e^{-\frac{R}{L} \cdot t}$$

$$\therefore e^{-\frac{R}{L} \cdot t} = 1 - \frac{1}{2}$$

$$\therefore -\frac{R}{L} \cdot t = \log \frac{1}{2}$$

$$= \log \frac{1}{2}$$

$$= \log 1 - \log 2$$

$$= O - \log 2$$

$$\not{-}\frac{R}{L} \cdot t = \not{-} \log 2$$

$$\therefore t = \frac{L \cdot \log 2}{R}$$

Check Point :

1) The equation of the eml in terms of current i for an electrical circuit having resistance R and a condensor of capacity C, in series is.

$$E = Ri + \int \frac{i}{c} \cdot dt$$

Find the current i at any time t , when

$$E = Eo \sin wt$$

Ans : $$i = \frac{WcEo}{\sqrt{1 + R^2C^2W^2}} \cos(wt - \phi) + c_1 \cdot e^{\frac{-t}{RC}}$$

$$\text{where } \phi = \tan^{-1}(RCW)$$

2) An electrical circuit contains an inductance of 5 henries and on resistance of 120 in series with an emf 120 sin (20t) Volts. Find current if it is zero when

$t = o$; at $t = 0.01$

Ans : $$\frac{20}{10144} \cdot \left[12 \sin(0.2) - 100 \cos \cdot (0.2) + 100.e^{\frac{-3}{125}}\right]$$

Newtom's Law of Cooling :

The law states that the rate at which the temp of a body changes is proportional to the difference between the instantenous temp of the body and the temp of the surrounding medium.

If Q is the instantenous temp of the body an Qo the temp of trhe surrounding then.

$$\frac{dQ}{dt} \alpha (Q - Qo)$$

$$\frac{dQ}{dt} = -k(Q - Qo)$$

where k is a constant and Q decreases as t increases i.e. $\frac{dQ}{dt}$ is negative hence negative sign is added.

Solved Examples :

1) The temperature of the air is 30^0C.and the substance cools from 100^0C to 70^0C in 15 minutes, find when the temperature will be 40^0C.

Soln.

Let us choose the unit of a time as a minute and Let T be the temp of the substance of any time t.

$\therefore$ By Newton's law of cooling

$$\frac{dT}{dt} = -k(T-30)$$

$$\therefore \frac{1}{T-30} \cdot dT = -kdt$$

Which is in variable separable form

Integrating both sides.

$$\int \frac{1}{(T-30)} \cdot dT = -kdt$$

$$\therefore \log(T-30) = -kt + c_1 \longrightarrow (1)$$

Initially $t = 0,\ T = 100$

$$\therefore \log(100 - 30 = 0 + c_1)$$

$$\therefore c_1 = \log 70$$

put value of c_1 in eq^n (1)

$$\therefore \log(T-30) = kt + \log 70$$

$$\therefore kt = \log 70 - \log(T-30) \longrightarrow (2)$$

when $t = 15,\ T = 70$

$\therefore$ from eq^n (2)

$$\therefore 15k = \log 70 - \log(70 - 30)$$

$$\therefore 15k = \log 70 - \log 40 \longrightarrow (3)$$

Dividing eq^n (2) by (3)

we get

$$\frac{t}{15} = \frac{\log 70 - \log(T-30)}{\log 70 - \log 40} \longrightarrow (4)$$

when $T = 40^0$ c , t = ?

From eq^n (4) -

$$\therefore \frac{t}{15} = \frac{\log 70 - \log(40 - 30)}{\log 70 - \log 40}$$

$$\therefore \frac{t}{15} = \frac{\log 70 - \log 10}{\log 70 - \log 40}$$

$$\therefore \frac{t}{15} = 3.48$$

$$\therefore t = 3.48 \times 15$$

$\therefore$ t = 52.20 minutes

$\therefore$ The temp will be 40^0 c after 52.20 minutes.

Q.2 If the temp. of the body drops from 100^0c to 60^0c in one minute when the temp. of the surrounding is 20^0c, What will be the temp. of the body at the end of second minute.

Soln :

Let us choose the unit of time as a minute and Let T be the temp. of the substance or body any time t.

$\therefore$ By Newton's law of cooling

$$\frac{dT}{dt} = -k(T-20)$$

$$\therefore \frac{1}{(T-20)} \cdot dT = -k \cdot dt$$

Which is in variable seperable form

Integrated both sides

$$\therefore \int \frac{1}{(T-20)} \cdot dT = -k \cdot \int dt + \text{constant}$$

$$\therefore \log(T-20) = -kt + c_1 \longrightarrow (1)$$

Initially at $t = 0, T = 100$

$\therefore \log(100 - 20) = 0 + c_1$

$\therefore c_1 = \log 80$

Put value of c_1 in eq^n (1)

$\therefore \log(T - 20) = -kt + \log 80$

$\therefore kt = \log 80 - \log(T - 20) \longrightarrow (2)$

When $t = 1, \quad T = 60^0$ c

$\therefore$ from (2)

$k = \log 80 - \log 40 \longrightarrow (3)$

Divide eq^n (2) by (3) we have

$4T - 80 = 80$

$T = 40^0$ c

Check Points :

(i) A body at temperature 100^0c is placed in a room whose temp is 20^0c and cools to 60^0c in 5 minutes. Find its temp. after a fruther interval of 3 minutes.

Ans :- 46.4^0 c.

9

Successive Differentiation

INTRODUCTION

In this chapter we shall study the methods of finding higher ordered derivatives for a given functional expression.

This is done in two stages.

Stage I : We shall eatablish some standard reesults and solve some problems using these results.

Stage II : We shall prove Lebnitz theorem and using it find higher order derivatives of given function

Notation:- Different notations used for derivatives of y=f(x) with respect to x are

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots \frac{d^ny}{dx^n}, \ldots\ldots \quad \text{(Due to Lebinitz)}$$

$$\dot{y}, \ddot{y}, \dddot{y} \quad \text{(Due to Newton)}$$

$$f(x), f(x), \ldots\ldots f^n(x) \quad \text{(Due to langravge)}$$

For convinence we also use notations

$y_1, y_2, \ldots\ldots y_n \ldots\ldots$ or $y', y'', y''' \ldots$ etc.

$(y_n)_0$ = value of n^{th} derivative of y at x = o

Stage (I)

Some standard results

(1)Let $y = e^{ax}$

$$y_1 = ae^{ax} \quad y_2 = a^2e^{ax} \ldots\ldots y_n = a^ne^{ax}$$

$$y = a^{mx}$$

$$y_1 = ma^{mx}(\log a), y_2 = m^2a^{mx}(\log a^2)\ldots\ldots$$

$$y_n = m^na^{mx}(\log a)^n$$

$$y = \sin(ax+b)$$

$$y_1 = a\cos(ax+b) = a\sin\left[\frac{\pi}{2} + (ax+b)\right]$$

(2) Let

(3)

$$y_2 = -a^2\sin(ax+b) = a^2\sin\left[2\frac{\pi}{2} + (ax+b)\right]\ldots\ldots$$

$$y_n = a^n\sin\left[(ax+b) + \frac{n\pi}{2}\right]$$

If a=l then

$$y = \sin(x+b) \quad \text{and} \quad y_n = \sin\left[(x+b) + \frac{n\pi}{2}\right]$$

Also if b=0 then y= sinx and $y_n = \sin\left[x + \frac{n\pi}{2}\right]$.

4)If $y = \cos(ax+b)$ then on similar lines $(m > n)$

$$y_n = a^n\cos\left(ax+b+\frac{n\pi}{2}\right)$$

$$y = (ax+b)^m \quad (m > n)$$

$$y_1 = ma(ax+b)^{m-1} \quad (m \text{ is integer})$$

$$y_2 = m(m-1)a^2(ax+b)^{m-2}$$

:

$$y_n = m(m-1)(m-2)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}$$

If m=n then $y_n = n(n-1)(n-2)\ldots\ldots1.a^n(ax+b)^0 = n!a^n$

If a=1, b=0, then $y=x^n$

$$\therefore y_n = n!$$

(6)

$$y = (ax+b)^{-m} \qquad (m \text{ is positive integer })$$

$$y_1 = (-m)a(ax+b)^{-m-1}$$

$$y_2 = (-m)(-m-1)a^2\,(ax+b)^{-m-2}\ldots\ldots\ldots\ldots$$

$$y_n = (-m)(-m-1)\ldots\ldots(-m-n+1)a^n\,(ax+b)^{-m-n}$$

$$= (-1)m(m+1)\ldots\ldots(m+n-1)a^n\,(ax+b)^{-m-n}$$

7)

$$y = \frac{1}{ax+b}$$

$$y_1 = \frac{-a}{(ax+b)^2} = -a(ax+b)^{-2}$$

$$y_2 = (-a)a(-2)(ax+b)^{-3} = \frac{(-1)^2 a^2 2!}{(ax+b)^3}\ldots\ldots.$$

: : :

$$y_n = \frac{(-1)^n n!\,a^n}{(ax+b)^{n+1}} \text{ if } a=1 \text{ then } y = \frac{1}{x+b^3}\ldots\ldots$$

$$\therefore \quad y_n = \frac{(-1)^n n}{(x+b)^{n+1}}$$

(8)

$$y = \log(ax+b)$$

$$y_1 = \frac{a}{(ax+b)}$$

(9) Let

and $\therefore r = \sqrt{a^2+b^2} \quad \alpha = \tan^{-1}\left(\frac{b}{a}\right)$

$$y_1 = e^{ax}[r\cos\alpha\sin(bx+c) + r\sin\alpha\cos(bx+c)]$$

$$= re^{ax}[\sin(bx+c+\alpha)]$$

$$y_1 = (a^2+b^2)^{1/2}\, e^{ax}\sin\left[bx+c+\tan^{-1}\left(\frac{b}{a}\right)\right]$$

similarly it can be proved that

$$y_2 = (a^2+b^2)^{2/2}\, e^{ax}\sin\left[bx+c+\tan^{-1}\left(\frac{b}{a}\right)\right]\ldots\ldots.$$

$$\therefore \quad y_n = (a^2+b^2)^{n/2}\, e^{ax}\sin\left[bx+c+\tan^{-1}\left(\frac{b}{a}\right)\right]$$

If $a=1, b=1, c=0$

$$y = e^{ax}\sin x$$

$$y_n = 2^{n/2}\, e^{x}\sin\left[x+\frac{n\pi}{4}\right]$$

$$y = e^{ax}\cos(bx+c)$$

$$y_n = (a^2+b^2)^{n/2}\, e^{ax}\cos\left((bx+c)+n\tan^{-1}\frac{b}{a}\right)$$

Type I

Problem 1

Find n^{th} derivatives of the following :

(i) $\sin^3 x$

(ii) $\cos x \ \cos 2x \ \ \cos 3x$

(iii) $y = e^x \cos x \cos 2x$

(iv) $y = e^{x\cos\alpha}\cos(x\sin\alpha)$

Sol.:

(i) Let $y = \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$

$$\sin 3x = 3\sin x - 4\sin^3 x$$

$$4\sin^3 x = 3\sin x - \sin 3x$$

$$\sin^3 x = \frac{1}{4}\cdot[3\sin x - \sin 3x]$$

Using the result for nth derivative of y=sin (ax+b) and nothing that nth derivative of sum or difference is sum or difference of nth derivatives, we get

$$y_n = \frac{1}{4}\left[3\sin\left(x + n\frac{\pi}{2}\right) - 3^n \sin\left(3x + \frac{n\pi}{2}\right)\right]$$

(ii) let

$$y = \cos x \cos 2x \cos 3x = \frac{1}{2}\cos 2x[\cos 4x + \cos 2x]$$

$$= \frac{1}{2}\cdot\cos 2x\cdot\cos 4x + \frac{1}{2}\cdot\cos 2x\cdot\cos 2x\left[\because\ c_A c_B = \frac{1}{2}(c_{A+B} + c_{A-B})\right]$$

$$= \frac{1}{2}\cdot\frac{1}{2}\cdot[\cos 6x + \cos 2x] + \frac{1}{2}\cdot\frac{1}{2}[\cos 4x + \cos 0]$$

$$= \frac{1}{4}[\cos x + \cos 2x] + \frac{1}{4} + [1 + \cos 4x] = \frac{1}{4}[\cos 6x + \cos 2x] + \frac{1}{4}[1 + \cos 4x]$$

$$= \frac{1}{4}[\cos 6x + \cos 4x + \cos 2x + 1]$$

$$y_n = \frac{1}{4}\left[6^n\cos\left(6x + \frac{n\pi}{2}\right) + 4^n\cos\left(4x + \frac{n\pi}{2}\right) + 2^n\cos\left(2x + \frac{n\pi}{2}\right)\right]$$

(iii) $y = e^x \cos x \cos 2x = \frac{e^x}{2}[\cos 3x + \cos x]$

$$= \frac{1}{2}[e^x \cos 3x + e^x \cos x]$$

[Using the result for nth derivative of y=e^{ax} cos (bx+c)

$$y_n = \frac{1}{2}\left[(10)^{n/2}\cdot e^x\cos(3x + \tan^{-1}3) + (2)^{n/2}\cdot e^x\cos\left(x + \frac{n\pi}{4}\right)\right]$$

(iv) $y = e^{x\cos\alpha}\cos(x\sin\alpha)$

[Here note $a = \cos\alpha,\ b = \sin\alpha,\ c = 0$]

$$\therefore\ y_n = (\cos^2\alpha + \sin^2\alpha)^{n/2} e^{x\cos\alpha}\cos[x\sin\alpha + n\tan^{-1}(\tan\alpha)]$$

$$\therefore\ y_n = e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha)$$

$\therefore\ y = (x-1)^n$

$\therefore\ y_n = (x-1)^{n-1}$

$y_1 = n(x-1)^{n-1}$

$y_2 = n(n-1)(x-1)^{n-2}$

:

$y_n = n!$

$\therefore y + y_1 + \frac{y_2}{2!} + \frac{y_3}{3!} + \dots \frac{y_n}{n!}$

Problem 2

Sol : if $y = \sin px + \cos px$, show that : $y_n = p^n[1 + (-1)^n \sin 2px]^{1/2}$

$$\therefore\ y = \sin px + \cos px$$

$$\therefore\ y_n = p^n \sin\left(px + \frac{n\pi}{2}\right) + p^n\cos\left(px + \frac{n\pi}{2}\right) \quad (\text{Results}: 3,4)$$

$$= p^n\left[\sin\left(px + \frac{n\pi}{2}\right) + \cos\left(px + \frac{n\pi}{2}\right)\right]$$

$$= p^n\left\{\left[\sin\left(px + \frac{n\pi}{2}\right) + \cos\left(px + \frac{n\pi}{2}\right)\right]^2\right\}^{1/2}$$

$$= p^n\left[1 + 2\sin\left(px + \frac{n\pi}{2}\right)\cdot\cos\left(px + \frac{n\pi}{2}\right)\right]^{1/2}$$

look at simplification : $(a+b) = \left[a+b) = [a+b)^2\right]^{1/2}$

$$= p^n[1 + \sin(2px + n\pi)]^{1/2} \quad [\because 2S_A C_A = S_{2A}]$$

$$= p^n[1 + (-1)^n \sin 2px]^{1/2} \quad [\because S_{A+B} = S_A C_B + C_A S_B]$$

$$\sin n\pi = 0$$

and $\cos nx = (-1)^n$

Problem 3

If y $\quad y = (x-1)^n$

Show that : $y + y_1 + \frac{y_2}{2!} + \frac{y_3}{3!} + \dots + \frac{y_n}{n!} = x^n$

Soln:

$$=(x-1)^n+(x-1)^{n-1}(1)+\frac{n(n-1)}{2!}(x-1)^{n-2}+\ldots\ldots\frac{n!}{n!}$$

$$=(x-1)^n+n(x-1)^{n-1}(1)+\frac{n(n-1)}{2!}(x-1)^{n-2}(1)^2+\ldots\ldots(1)^n$$

$$=[(x-1)+1]^n$$

$$=(x)^n$$

$$\Big[(a+b)^n=a^n+na^{n-1}b \quad \text{(Note: See Binomial expansion)}$$

$$+\frac{n(n-1)}{2!}a^{n-2}b^2+\ldots\ldots b^n$$

Here $a=x-1, b=1\Big]$

Type II

Find n^{th} derivatives by method of fraction :

Problem I

Find n^{th} derivative of

(i) $y=\frac{x}{(x-1)(x-2)(x-3)}$

Soln :

(i) using method of partial fraction

$$y=\frac{x}{(x-1)(x-2)(x-3)}$$

$$\frac{x}{(x-1)(x-2)(x-3)}=\frac{A}{(x-1)}+\frac{B}{(x-2)}+\frac{C}{(x-3)}$$

$$\frac{x}{(x-1)(x-2)(x-3)}=\frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\therefore x=A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)$$

Put $x=1$ Put $x=2$ Put $x=3$

$$\therefore A=\frac{1}{2} \quad B=-2 \quad C=\frac{3}{2}$$

$$\therefore y=\frac{1}{2}\frac{1}{(x-1)}-\frac{2}{(x-2)}+\frac{3}{2}\frac{1}{(x-3)}$$

$$\therefore y_n=\frac{1}{2}\frac{(-1)^n n!}{(x-1)^{n+1}}-2\frac{(-1)^n\cdot n!}{(x-2)^{n+1}}+\frac{3}{2}\cdot\frac{n!}{(x-3)^{n+1}}$$

Type III : Using Complex Numbers

Problem 1

Find n^{th} derivative of $y=\frac{1}{x^2+a^2}$

Soln.:

We have

$$y=\frac{1}{(x-ai)(x+ai)}$$

By partial fractions,

$$y=\frac{1}{2ai(x-ia)}-\frac{1}{2ai(x+ia)}$$

Using result (7) $y_n=\frac{1}{2ai}\frac{(-1)^n n!}{(x-ia)^{n+1}}-\frac{1}{2ai}\frac{1}{(x+ia)^{n+1}}$

$$=\frac{(-1)^n\cdot n!}{2ia}\left[\frac{1}{(x-ia)^{n+1}}-\frac{1}{(x+ia)^{n+1}}\right]$$

To eliminatei, we substitute

$$x+ia=-r\cdot e^{i\theta}\ \therefore x-ia=r\cdot e^{-i\theta}$$

(1) Becomes,

$$=\frac{(-1)^n\cdot n!}{2ia}\left[\frac{i}{(re^{-\theta})^{n+1}}-\frac{1}{(re^{+\theta}i)^{n+1}}\right]$$

$$=\frac{(-1)^n\cdot n!}{2ia\cdot r^{n+1}}\left[e^{i(n+1)\theta}-e^{-i(n+1)\theta}\right]$$

$$=\frac{(-1)^n\cdot n!}{ar^{n+1}}\left[\frac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{2i}\right] \qquad \sin=\frac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{2i}$$

$$\therefore \qquad y_n=\frac{(-1)^n\cdot n!}{ar^{n+1}}\cdot\sin[(n+1)\theta]$$

$$r=\sqrt{a^2+x^2},\theta=\tan^{-1}\frac{a}{x}$$

Problem 2

Find n^{th} derivative of $y = \frac{x}{x^2 + a^2}$

Soln.:

We have $y = \frac{x}{(x-ia)(x+ia)}$

$$= \frac{ia}{(x-ia)(2ai)} + \frac{-ia}{(x+ia)(-2ai)}$$

$$\therefore y = \frac{1}{2}\left[\frac{1}{(x-ia)} + \frac{1}{(x+ia)}\right]$$

$$\therefore y_n = \frac{1}{2}\left[\frac{(-1)^n \cdot n!}{(x-ia)^{n+1}} + \frac{(-1)^n \cdot n!}{(x+ia)^{n+1}}\right]$$

Let $x + ia = re^{i\theta}, x - ia = re^{-i\theta}$

$$\therefore \ r = \sqrt{x^2 + a^2},\ \theta = \tan^{-1}\frac{a}{x}$$

From (1) $y_n = \frac{(-1)^n \cdot n!}{2}\left[\frac{1}{r^{n+1} \cdot e^{-i(n+1)\theta}} + \frac{1}{r^{n+1} \cdot e^{i(n+1)\theta}}\right]$

$$= \frac{(-1)^n \cdot n!}{r^{n+1}}\left[\frac{e^{i(n+1)\theta} + e^{-i(n+1)\theta}}{2}\right]$$

$$\therefore y_n = \frac{(-1)^n \cdot n!}{r^{n+1}} \cos\left[(n+1)^{\theta}\right]$$

$$r = \sqrt{x^2 + a^2},\ \tan^{-1}\frac{a}{x}$$

Problem 4

Find n^{th} derivative of $\tan^{-1} x$

Sol.: $y = \tan^{-1} x$

$$y_1 = \frac{1}{x^2 + 1} = \frac{1}{(x-i)(x+i)}$$

$= \frac{1}{2i}\left[\frac{1}{(x-i)} - \frac{1}{(x+i)}\right]$ $\because$ is $(n-1)^{th}$ derivatives of y_1, we have

$$y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(X-i)^n} - \frac{(-1)^{n-1}(n-1)!}{(X+i)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(X-i)^n} - \frac{1}{(X+i)^n}\right]$$

Let $x + 1 = re^{i\theta}, \quad x - i = re^{-i\theta}$

$$\therefore \quad r = \sqrt{1 + x^2}, \theta = \tan^{-1}\frac{1}{x}$$

$$\therefore \ y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{(re^{-i\theta})^n} - \frac{1}{(re^{i\theta})^n}\right]$$

$$y_n = \frac{(-1)^{n-1} \cdot (n-1)!}{r^n}\left[\frac{e^{in\theta} - e^{-in\theta}}{2!}\right]$$

Where $= \frac{(-1)^{n-1}(n-1)!}{r^n} \cdot \sin(n\theta)$

$$r = \sqrt{x^2 + 1} \qquad \theta = \tan^{-1}\frac{1}{x}$$

Problem 5

Find n^{th} derivative of :

(i) $\cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$

Soln. :

$$\cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$$

$$y = \cos^{-1}\left(\frac{x - x^{-1}}{x + x^{-1}}\right)$$

$$= \cos^{-1}\left(\frac{x^2 - 1}{x^2 + 1}\right)$$

$$x = \tan\theta$$

$$y = \cos^{-1}(-\cos 2\theta)$$

$$= \cos^{-1}[\cos(\pi + 2\theta)]$$

$$= \pi + 2\theta$$

$= \pi + 2\tan^{-1} x$, from previous result.

$$y_n = 2\frac{(-1)^{n-1}(n-1)!}{r^n} \cdot \sin n\alpha$$

Where $r = \sqrt{1+x^2}$ $\quad \alpha = \tan^{-1}\frac{1}{x}$

Type (IV)

Problem 1:

If y=sinx (sinx),

Show that : $$\frac{d^2y}{dx^2} + \tan x \cdot \frac{dy}{dx} + y\cos^2 x = 0$$

$$y = \sin(\sin x)$$

$$\therefore \quad \frac{dy}{dx} = \cos(\sin x)\cos x$$

$$\frac{d^2y}{dx^2} = -\sin(\sin x)\cos^2 x - \sin x \cdot \cos(\sin x)$$

$$\therefore \quad \frac{d^2y}{dx^2} + \tan x\frac{dy}{dx} + y\cos^2 x$$

$$= \ -\sin(\sin x)\cos^2 x - \sin x \cdot \cos(\sin x)$$

$$+ \tan x.\cos x.\cos(\sin x) + \cos^2 x\sin(\sin x) = 0$$

Problem 2:

If $p^2 = a^2\cos^2\theta + b^2\sin^2$, prove that

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Sol.:

Diff. given relation twice,

$$2p\frac{dp}{d\theta} = \text{-}a^2 2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta$$

$$\therefore \quad p\frac{dp}{d\theta} = \left(b^2 - a^2\right)\sin\theta\cos\theta \text{---------------}(1)$$

$$\frac{d^2p}{d\theta^2} + \left(\frac{dp}{d\theta}\right)^2 = \left(b^2 - a^2\right)\left(\cos^2\theta - \sin^2\theta\right)\text{...................}(2)$$

$$\therefore \quad p\frac{d^2p}{d\theta^2} + \frac{\left(b^2 - a^2\right)\sin^2\theta\cos^2\theta}{p^2} = \left(b^2 - a^2\right)\left(\cos^2\theta - \sin^2\theta\right)$$

$$\therefore \quad p^3\frac{d^2p}{d\theta^2} + p^2\left(b^2 - a^2\right)\left(\cos^2\theta - \sin^2\theta\right) - \left(b^2 - a^2\right)\sin^2\theta \cdot \cos^2\theta$$

$$= \left(b^2 - a^2\right)\left[p^2\left(\cos^2\theta - \sin^2\theta\right) - \left(b^2 - a^2\right)\sin^2\theta \cdot \cos^2\theta\ \right]$$

$$= \left(b^2 - a^2\right)\left[\left(a^2\cos^2\theta + b^2\sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right) - \left(b^2 - a^2\right)\sin^2\theta \cdot \cos^2\theta\right]$$

$$= \left(b^2 - a^2\right)\left[a^2\cos^4\theta - b^2\sin^4\theta\right]$$

$$= a^2b^2\left(\cos^4\theta + \sin^4\theta\right) - \left(a^4\cos^4\theta + b^4\sin^4\theta\right)$$

$$= a^2b^2\left[\left(\cos^2\theta + \sin^2\theta\right)^2 - 2\sin^2\theta \cdot \cos^2\theta\right] - \left(a^4\cos^4\theta + b^4\sin^4\theta\right)$$

$$= a^2b^2 - \left(a^2\cos^2\theta + b^2\sin^2\theta\right)$$

$$= a^2b^2 - p^4$$

$$\therefore \quad p^3\frac{d^2p}{d\theta^2} = a^2b^2 - p^4$$

$$\therefore \quad p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$$

Problem 4 :

If $$y = \sin^{-1}\left(\frac{1+2\sin x}{2+\sin x}\right)$$

Then show that **:** $$\frac{dy}{dx} = \frac{\sqrt{3}}{2+\sin x}$$

Sol.:

We have
$$\sin y = \frac{1+2\sin x}{2+\sin x} = \frac{2\sin x + 4 - 3}{2+\sin x} = 2 - \frac{3}{2-\sin x}$$

$$\therefore \cos y\frac{dy}{dx} = \frac{3\cos x}{(2+\sin x)^2}$$

Now $$\cos y = (1-\sin y)^{1/2}$$

$$= \left[1 - \frac{(1+2\sin x)^2}{(2+\sin x)^2}\right]^{1/2}$$

$$= \left[\frac{(2+\sin x)^2 - (1+2\sin x)^2}{(2+\sin x)^2}\right]^{1/2}$$

$$= \frac{\sqrt{3}\cos x}{2+\sin x}$$

$$\frac{dy}{dx} = \frac{3\cos x}{(2+\sin x)^2}\cdot\frac{1}{\cos y}$$

$$= \frac{3\cos x}{(2+\sin x)^2}\cdot\frac{(2+\sin x)}{\sqrt{3}\cos x}$$

$$= \frac{\sqrt{3}}{2+\sin x}$$

Check Points :

1) If y=xe^y then show that :

$$(1-y)\frac{d^2y}{dx^2} = (2-y)\left(\frac{dy}{dx}\right)^2$$

$y = x^2 e^x \cos x$

$u = x^2, \quad v = e^x \cos x$

$$y_n = u v_n + nc_1 u_1 v_{n-1} + nc_2 u_2 v_{n-2} + \ldots\ldots..$$

$$= x^2\left[2^{n/2} e^x \cos x\left(x + \frac{n}{4}\right)\right] + n(2x)\left[2^{\frac{n-1}{2}} e^x \cos\left(x + \frac{(n-1)}{4}\right)\right]$$

$$+ \frac{n(n-1)}{2!}(2)\cdot\left[2^{\frac{n-2}{2}}\cdot e^x \cos\left(x + \frac{(n-2)}{4}\right)\right]$$

2) If $y = (1-x^2)^{1/2}\cdot\sin^{-1}x$, then show that :

$$(1-x^2)\frac{d^2y}{dx^2}\times\frac{dy}{dx} + 2x + y = 0$$

[Hint : sub $x = \sin\theta$. $\therefore y = \theta\cos\theta$

find $\frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$........]

3) If $y = (\sin^{-1}x)^2$ then show that:

$$(1-x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 2 = 0$$

[Hint: $\frac{dy}{dx} = 2\sin^{-1}x\cdot\frac{1}{\sqrt{1+x^2}}$

$\therefore (1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y$, differentiate again]

4) If $y = xe^y$, then show that :

$$(1-y)\frac{d^2y}{dx^2} = (2-y)\left(\frac{dy}{dx}\right)^2$$

[Hint : Take log]

5) If $my = \sin(x+y)$

show that $y_2 = -y(1+y_1)^3$, m is constant

6) Find nth derivative of $\sin^5 x\cdot\cos^3 x$

Problems

Type (I) : In this type of problem we have to choosen one function as u and the other as v. If there is a polynomial in x then that is to be chosen as u and then apply Leibnitz theorem.

Problem I

Find nth derivative of

Soln.:

Let

Using standard result number

$$y_n = 2^{n/2} e^x \cos\left(x + \frac{n\pi}{4}\right).$$

Here $y = u\cdot v$

By Leibnitz theorem

Problem 2

If f(x)=tanx then

show that : $f^{n}(0)-{}^{n}c_{2}f^{n-2}(0)^{n}+{}^{n}c_{4}f^{n-4}(0)......=\sin\left(\frac{n\pi}{2}\right)$

Soln.: $\cos x\cdot f(x)=\sin x$

Taking n^{th} derivatives both sides to the left side we apply leibnitz theorem and to the right we use standard formula for n^{th} deriative of sinx, we get,

$$\cos x.f^{n}(x)+{}_{n}C_{1}-(\sin x)f^{n-1}(x)+{}^{n}C_{2}(-\cos x)f^{n-2}(x)+......=\sin\left(\frac{n\pi}{2}\right)$$

putting x=0 on both sides, we get,

$$f^{n}(0)+n_{C_2}f^{n-2}(x)+n_{C_4}f^{n-4}(0)+......=\sin\left(\frac{n\pi}{2}\right)$$

Problem 3

If $y=\frac{\log x}{x}$ then show that :

$$y_n=\frac{(-1)^n\cdot n!}{x^{n+1}}\left[\log x-\left(1+\frac{1}{2}+\frac{1}{3}+.....+\frac{1}{n}\right)\right]$$

Soln. : Let $u=\log x,\ v=\frac{1}{x}$

By standard results we have

$$u_n=\frac{(-1)^{n-1}(n-1)n}{x^n}$$

$$v_n=\frac{(-1)^n\cdot n!}{x^{n+1}}$$

We have y=u, v

By Leibnitz theorem

$$y_n=uv_n+{}^{n}C_1u_1v_{n-1}+{}^{n}C_2u_2v_{n-2}+.......+u_nv$$

$$=\log x\cdot\frac{(-1)^n.n!}{x^{n+1}}+n\frac{1}{x}\frac{(-1)^{n-1}(n-1)n}{x^n}+\frac{n(n-1)}{2!}\times$$

$$\left(-\frac{1}{x^2}\right)\times\frac{(-1)^{n-2}(n-2)n}{x^{n-1}}+.....+\frac{(-1)^{n-1}(n-1)n}{x^n}\frac{1}{x}$$

$$\left[\because(-1)^{n-1}=(-1)^n\cdot(-1)^{-1}=-(-1)^n\right]$$

$$\therefore\ y_n=\frac{(-1)^n.n!}{x^{n+1}}\left[\log x-\left(1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{n}\right)\right]$$

Problem 4

If $I_n=\frac{d^n}{dx^n}\left(x^n\cdot\log x\right)$

then show that

(i) $I_n={}_{n}1_{n-1}+(n-1)!$ and

(ii) $I_n=n!\left[\log x+1+\frac{1}{2}+.....+\frac{1}{n}\right]$

Soln.:

(i) We have

$$\therefore\quad I_n=\frac{d^n}{dx^n}\left(x^n\cdot\log x\right)=\frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}\left(x^n\log x\right)\right]$$

$$=\frac{d^{n-1}}{dx^{n-1}}\left[nx^{n-1}\log x+x^{n-1}\right]$$

$$=n\frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1}\cdot\log x\right]+\frac{d^{n-1}}{dx^{n-1}}\left[x^{n-1}\right]$$

$$\therefore\quad I_n=n1_{n-1}+(n-1)!$$

$\left[\text{if } n^{th}\text{ derivative of } x^n \text{ is } n! \text{ therefore } (n-1)^{th}\text{ derivative of } x^{n-1}\text{ is } (n-1)!\right]$

ii dividing (1) on both sides by n!

$$\frac{I_n}{n!}=\frac{nI_{n-1}}{n!}+\frac{(n-1)!}{n!}$$

i.e. $\frac{I_n}{n!}=\frac{I_{n-1}}{(n-1)!}+\frac{I}{n}$

Replacing n by $(n-1)(n-2)$............3,2,1 we get

$$\frac{1_{(n-1)}}{(n-1)!}=\frac{1_{n-2}}{(n-2)!}+\frac{1}{(n-1)}$$

$$\frac{I_{(n-2)}}{(n-2)!} = \frac{I_{n-3}}{(n-3)!} + \frac{1}{n-2}$$

:

$$\frac{I_2}{2!} = \frac{I_1}{1} + \frac{1}{2}$$

$$\frac{I_1}{1!} = \frac{I_0}{0!} + 1$$

Adding all the results columnwise we get,

$$\frac{I_n}{n!} + \frac{I_{n-1}}{(n-1)!} + \frac{I_{n-2}}{(n-2)!} + ... + \frac{I_2}{2!} + \frac{I_1}{1!}$$

$$= \frac{I_{n-1}}{(n-1)!} + \frac{I_{n-2}}{(n-2)!} + ... + \frac{I_1}{2!} + \frac{I_0}{0!}$$

$$+ \left[\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + ... + \frac{1}{2} + 1\right]$$

cancelling common terms on both sides and noting that

$$I_0 = 0^{th} \text{ derivative of } x^0 \log x$$

$$= \log x$$

and $0! = 1$ we get

$$I_n = n!\left[\log x + 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n}\right]$$

Problem 5

By forming in two different ways the n^{th} derivative of x^{2n} show that :

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2 - 2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2,2^2,3^2} + ... \frac{(2_n)!}{(n!)^2}$$

Soln.:

Step 1: We have

$$y = x^{2n} \qquad \text{Standard formula}$$

$$\therefore \quad y_n = (2n)(2n-1)........(2n-n+1)x^{2n-n}$$

$$= \frac{[(2n)(2n-1)........(n+1)][n(n-1)......3,2,1]x^n}{[n(n-1)......3,2,1]}$$

$$= \frac{(2_n)!}{n!}x^n$$

Step 2: Again $y = x^{2n} = x^n . x^n$

We apply Leibtnitz theorem to find y_n

$$u = x^n, v = x^n$$

$$u_n = n!, \; v_n = n!$$

$$\therefore \; y_n = x^n \cdot n! + {}^nC_1\left(nx^{n-1}\right)(n!\,x) + \left(n(n-1)x^{n-2}\left(n!\frac{x^2}{2}\right) + ... + x^n.n!\right) \text{ see note below}$$

$$\therefore \; y_n = x^n.n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{(2!)^2} + \frac{n^2(n-1)^2(n-2)^2}{(3)^2} + ... + \right]$$

$$= x^n.n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + ... + \right]$$

From (1) and (2)

$$\frac{(2n)!}{n!}x^n = x^n.n!\left[1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + ...\right]$$

$$= 1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + ... = \frac{(2_n)!}{(n!)^2}$$

Note : The n^{th} derivative of x^n is n! but $(n-1)^{th}$ derivative of x^n is not (n-1)! but n! .x.

To prove that this use the formula No.5 and put m = (n-1).

Type (II)

In this type of problems we (generally) proceed according to the following flow-diagram:

First express y in terms of x

↓

Differentiate both sides with respect to x and simplify

↓

Again Differentiate both sides and simplify

↓

Then apply Leibnitz theorem term by term and simplify to get the result.

Problem 1

If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$

Show that : $(x^2-1)y_{m+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Soln

$$y^{1/m} + y^{-1/m} = 2x$$

$$\therefore \quad y^{1/m} + \frac{1}{y^{1/m}} = 2x$$

$$\left(y^{1/m}\right)^2 - 2xy^{\frac{1}{m}} + 1 = 0$$

is a quadratic equation in $y^{1/m}$

$$\therefore \quad y^{1/m} = \frac{2x \pm \sqrt{4x^2-4}}{2}$$

$$\therefore \quad y^{1/m} = x + \sqrt{x^2-1} \qquad \text{(neglecting negative sign)}$$

$$\therefore \quad y = \left(x + \sqrt{(x^2-1)}\right)^m$$

$$\therefore \quad y_1 = m\left(x + \sqrt{(x^2-1)}\right)^{m-1} \cdot \left(1 + \frac{2x}{2\sqrt{(x^2-1)}}\right)$$

$$\therefore \quad y_1 = m\left(x + \sqrt{(x^2-1)}\right)^{m-1} \cdot \frac{\left(x + \sqrt{(x^2-1)}\right)}{\sqrt{(x^2-1)}}$$

$$\therefore \sqrt{x^2-1} - y_1 = m\left(x + \sqrt{(x^2-1)}\right)^m$$

$$= m \cdot y$$

$$\therefore (x^2-1)y^2{}_1 = m^2y^2$$

Differentiating both the sides with respect to x, we get,

$$2(x^2-1)y_1y_2 + 2xy^2{}_1 - 2m^2yy_1 = 0$$

$$\therefore (x^2-1)y_2 + xy_1 - m^2y = 0$$

Applying Leibnitz term by term to find n^{th} derivative we get,

$$\left[(x^2-1)y_{n+2} + n(2x)y_{n+1} + \frac{n(n-1)}{2!}(2)y_n\right] + \left[xy_{n+1} + n(1)y_n\right] - m^2y_n = 0$$

$$\therefore (x^2-1)y_{n+2} + x(2n+1)y_{n+1} + (n^2-m^2)y_n = 0$$

Note : If we consider negative sign, we shall get the same result.

Problem 2

If we $\cos^{-1}\left(\frac{y}{b}\right) = \log\left(\frac{x}{n}\right)^n$ then

Show that **:** $x^2y_{n+2} + (2n+1)xy_{n+1} + 2n^2y_n = 0$

Soln. **:**

We have

$$\cos^{-1}\frac{y}{b} = n\log(\log x - \log n)^n$$

$$\therefore \quad y = b\cos[n\log x - n\log n]$$

$$\therefore \quad y_1 = -b\sin[n\log x - n\log n]\left(\frac{n}{x}\right)$$

$$\therefore \quad xy_1 = -nb\sin[n\log x - n\log n]$$

differentiating both the sides with respect to x

∴ $xy_2 + y_1 = -nb\cos[n\log x - n\log n]\cdot\left(\frac{n}{x}\right)$

∴ $x^2y_2 + xy_1 = -n^2[b\cos(n\log x - n\log n)]$

∴ $= -n^2y$

$x^2y_2 + xy_1 + n^2y = 0$

Applying Leibnitz theorem term by term to differentiate n times, we get,

$$\left[x^2y_{n+2} + n2xy_{n+1} + \frac{n(n-1)}{2!}(2)y_n\right] + [xy_{n+1} + n(1)y_n] - n^2y_n = 0$$

$$x^2y_{n+2} + x(2n+1)y_{n+1} + 2n^2y_n = 0$$

Problem 3

If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then show that

$$(1-x^2)y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2y_n = 0$$

Soln. :

We have $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$

∴ $(1-x^2)y^2 = (\sin^{-1}x)^2$

Differentiating with respect to both sides,

$$(1-x^2)2yy_1 - 2xy^2 = 2\frac{\sin^{-1}x}{\sqrt{1-x^2}} = 2y$$

∴ $(1-x^2)y_1 - xy = 1$

Differentiating with respect to x

$$(1-x^2)y_2 - 2xy_1 - xy_1 - y = 0$$

$$(1-x^2)y_2 - 3xy_1 - xy_1 - y = 0$$

Applying Leibnitz term by term, we get

$$\left[(1-x^2)y_{n+2} + n(-2x)y_{n-1} + \frac{n(n-1)}{2!}(-2)y_n\right] - 3[xy_{n+1} + n\cdot 1\cdot y_n] - y_n = 0$$

∴ $(1-x^2)y_{n+2} - x(2n-3)y_{n+1} - (n+1)^2y_n = 0$

Problem 4 $y = \sec^{-1}x$

then show that

$$x(x^2-1)y_{n+2} + [(2+3n)x^2 - n + 1]y_{n+1} + n(3n+1)xyn + n^2(n-1)y_n - 1 = 0$$

Soln.: $y = \sec^{-1}x$

Differentiating with respect to x

$$y_1 = \frac{1}{x\sqrt{x^2-1}}$$

∴ $x^2(x^2-1)y^2{}_1 = 1$

i.e. $(x^4 - x^2)y^2{}_1 = 1$

Differentiating with respect to x

$$(4x^3 - 2x)y^2{}_1 + (x^4 - x^2)2y_1y_2 = 0$$

i.e. $(2x^2-1)y_1 + (x^3 - x)y_2 = 0$

i.e. $(x^3 - x)y_2 + (2x^2-1)y_1 = 0$

Differentiating term by term n times using Leibnitz theorem, we get,

$$\left[(x^3-x)y_{n+2} + n(3x^2-1)y_{n+1} + \frac{n(n-1)}{2!}(6x)y_n\right]$$

$$+3\left[\frac{n(n-1)(n-2)}{3!}(6)y_{n-1}\right]$$

$$+\left[(2x^2-1)y_{n+1} + n(4x)y_n + \frac{n(n-1)}{2!}(4)y_{n-1}\right] = 0$$

i.e. $(x^3-x)y_{n+2} + n(3x^2-1)(2x^2-1)y_{n+1} + [3n(n-1)x + 4nx]y_n$

$+ [n(n-1)(n-2) + 2n(n-1)]y_{n-1} = 0$

i.e. $x(x^2-1)y_{n+2} + [(2+3n)x^2 - (n+1)]y_{n+1}$

$$+n(3n+1)xy_n+n^2(n-1)y_{n-1}=0$$

Problem 5

If $y=\tan^{-1}x$, then S.T.

$(x^2+1)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0$

and also show that : $y_n(0)$ is 0, $(n-1)!$ or 4r + 3 respectively

Soln.:

Step I : $y=\tan^{-1}x$

$$\therefore \quad y_1=\frac{1}{1+x^2}$$

$$\therefore \quad (x^2+1)y_1=1$$

Applying Leibtnitz theorem to differentiate n times, we get,

$$\left[(x^2+1)y_{n+1}+n(2x)y_n+\frac{n(n-1)}{2!}(2)y_{n-1}\right]=0$$

i.e. $(x^2+1)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0$.......................(1)

Step (II) :

Now, $y_1(0)=1$

And $y_2(0)=\dfrac{-2x}{(1+x^2)^2}$

putting $y_2(0)=0$

$\therefore \quad n=2,3,4,5,6$ in (1)

$y_3=-2=-(-2)!$

$y_4=0$

$y_5=4!$

$y_6=0$

$y_7=-6!$

:

$\therefore \quad y_n(0)=0 \; n=2r$

$y_n(0)=(n-1)! \text{ if } n=4r+1$

Check Point :

1) If $y=\sin(m\sin^{-1}x)$ then show that

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2-m^2)y_n=0$

[Hint : Find y_1 then

$(1-x^2)y_1^2=m^2(1-y^2)$ and again differentiate and apply L. theorem]

2) If $y=e^{a\sin^{-1}x}$ then show that

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0$

3) If $y=a\cos(\log x)-b\sin(\log x)$

then show that

$x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$

[Hint : $x^2y_2+xy_1+y=0\rightarrow$ apply Leibnitz theorem]

4) If $y=(\sin^{-1}x)^2$ then show that :

$(1-x^2)y_{n+2}(2n+1)-xy_{n+1}-n^2y_n=0$

5) If $x=\tan(\log y)$ then show that :

$(1-x^2)y_{n+2}+[2(n+1)x-1]y_{n+1}+n(n+1)y_n=0$

[We have $y=e^{\tan^{-1}x}$.....]

6) If $y=\sin[\log(x^2+x+1)]$, prove that

$(x+1^2)y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0$

[Hint : Find y_1, simplify, find y_2 simplify and apply leibnitz theorem]

$\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}$ or f_x.

10

Partial Differentiation

Introduction

So far, we have been concerned with a functions of a variable, but in many problems in science and mathematics we have to deal with functions of two or more independent variables.

e.g. the lift L of an aeroplane wing is a function of three independent variables : A, the area of the wing, V, the speed at which the wing is moving; and P the density of the air. The law is $L = Akv^2p$

In the language of mathematics, if variable u has one definite value for any given values of x,y,z then u is defined as a function of x,y,z.We represent it as u= f(x,y,z)

Note that u is independent variable and x,y,z are indepedent variables. This relation is written as - $u \rightarrow x, y, z$

Partial Differential Coefficients :

The partial derivative of u= f(x,y,z) with respect to x is the oridnary derivative of u with respect to x when y and z are regarded as constant. It is denoted by

(To be pronounced as dabba u by dabba x)

Thus, $\frac{\partial u}{\partial x}, \frac{\partial f}{\partial x}$ or $f_x = \lim_{x \to o} \frac{f(x+h, y, z) - f(x, y, z)}{h}$

Similarly when we differentiate u with respect to y we keep x and z constant and so on .

In general, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$ are also functions of $x, y, z,$ so we can obtain higher ordered partial derivatives of $u = f(x, y, z)$

e.g. $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x^2}=f_{xx}=\frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial^2 u}{\partial y\partial x}=f_{yx}=\frac{\partial^2 f}{\partial y\partial x}$$

and $$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial^2 u}{\partial x\partial y}=f_{xy}=\frac{\partial^2 f}{\partial x\partial y}$$ and so on.

Note :

In general , $\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial^2 u}{\partial y\partial x}$

9.3 RULES OF PARTIAL DIFFERENTIATION :

(1) Let,u v be functions of x,y,z Then

$$\frac{\partial}{\partial x}(u\pm v)=\frac{\partial u}{\partial x}\pm\frac{\partial v}{\partial x}$$

(2) $$\frac{\partial}{\partial x}(uv)=u\frac{\partial v}{\partial x}+v\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial x}(kv)=k\frac{\partial v}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right)=\frac{v\frac{\partial u}{\partial x}-u\frac{\partial v}{\partial x}}{v^2}$$

$$\frac{\partial}{\partial x}\left(\frac{k}{v}\right)=-\frac{k}{v^2}\cdot\frac{\partial v}{\partial x}$$

(II) CHAIN RULES

Chain- rules are to be developed by drawing flow- diagrams.

Study this point carefully.

(1) Let $u=f(x,y,z)$ and $x=\phi_1$ $t=\phi_2(t)$, $z=\phi_3(t)$

[i.e. u is a function of x,y,z and x,y,z each is a function of only variable t]

Thus ,

$$\therefore \quad \frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial u}{\partial z}\cdot\frac{dz}{dt}$$

($\because$ is a function of only variable t,

$\therefore$ we wirte total derivative $\frac{du}{dt}$ and not $\frac{\partial u}{\partial t}$)

e.g. if $u=x^2+y^2+z^2, x=t, y=t^2, z=t^3$

then $u\rightarrow x,y,z\rightarrow t$

$$\frac{\partial u}{\partial t}=(2x)\cdot 1+(2y)(2t)+(2z)3t^2$$

(2)

(3) If $u=f(x,y,z),\ x=\phi_1(r,s),$

$y=\phi_2(r,s),$

$z=\phi_3(r,s),$

then the flow diagram becomes,

i.e. $u\rightarrow x,y,z\rightarrow r,s$

If we want $\frac{\partial u}{\partial s}$ then it is given by

$$\frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}$$

e.g. $u=x^2+y^2+z^2, x=r+s+t,\ y=s^2+t^2, z=t^3$

then $\frac{\partial u}{\partial s}=2x\cdot 1+2y\cdot 2s+2z\cdot 0=6x+4ys$

(III) TOTAL DIFFERENTIATION

In Partial differentiation of a function of two or more variables , only one variable varies. But in total differentiation, increments are given in all the variables.

Let $\quad z = f(x,y)$

Let ∂z be the increment in z corresponding to the increments ∂x and ∂y in x and y respectively

Replace ∂ by δ only

Then $\quad z + \delta z = f(x + \partial x, y + \partial y)$

$\partial z = f(x + \partial x, y + \partial y) - f(x,y)$

$\therefore \quad \partial z = f(x + \partial x, y + \partial y) - f(x, y + \partial y) + f(x, y + \partial y) - f(x,y)$

or $\quad \partial z = \left[\dfrac{f(x+\partial x, y+\partial y) - f(x, y+\partial y)}{\partial x}\cdot \partial x\right] + \left[\dfrac{f(x, y+\partial y) - f(x,y)}{\partial y}\right]$

∂y

Taking limits as $\partial x \to 0, \partial y \to 0$ we get $\partial z = \dfrac{\partial f}{\partial x}\cdot dx + \dfrac{\partial f}{\partial y}\cdot dy$

dz is called as total differential of z

Let us see some Corollaries:

(1) Let $u = f(x,y,z)$ and $x = \phi_1(t), y = \phi_2(t), z = \phi_3(t)$

[i.e. u is function of x,y,z and x,y,z each is a function of only one variable t.]

Thus,

$u = \to x,y,z \to t$

$\therefore \quad \dfrac{du}{dt} = \dfrac{\partial u}{\partial x}\cdot\dfrac{dx}{dt} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dt} + \dfrac{\partial u}{\partial z}\cdot\dfrac{dz}{dt}$ ($\because$ u is a function of only one variable t,

$\therefore$ We write total derivative $\dfrac{du}{dt}$ and not $\dfrac{\partial u}{\partial t}$)

e.g. If $u = x^2 + y^2 + z^2, y = t^2, z = t^3$

then $u \to x,y,z \to t$

$$\frac{du}{dt} = (2x)\cdot 1 + (2y)(2t) + (2z)3t^2$$

(2)

Let $u = f(x,y)$ and $\phi(x,y) = 0$

$\because \phi(x,y) = 0$, y can be regarded as a function of x and hence flow - diagram is

$u \to x, y \to x$

$\therefore \quad \dfrac{du}{dx} = \dfrac{\partial u}{\partial x}\cdot 1 + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}$

e.g. if $\quad u = x^2 + y^2$

and $\quad x^3 + y^3 + 3xy = 4 \quad$ then to find $\dfrac{du}{dx}$

$\because \quad x^3 + y^3 + 3xy = 4$

Differentiate with respect to x,

$$3x^2 + 3y^2\frac{dy}{dx} + 3y + 3x\frac{dy}{dx} = 0$$

$$\therefore \quad \frac{dy}{dx} = -\frac{(x^2 + y)}{(x + y^2)}$$

and $\quad \dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx} = 2x + 2y\left[-\dfrac{x^2 + y}{x + y^2}\right]$

$$= \frac{2x(x + y^2) - 2y(x^2 + y)}{(x + y^2)} = \frac{2[x^2 - y^2 \neq xy^2 - x^2 y]}{(x + y^2)}$$

(3) If $f(x,y) = 0$ then to find $\dfrac{dy}{dx}$

[This result is a special case of result (4)].

Let $u = f(x,y)$ and $f(x,y) = 0$

$\therefore \quad u \to x, y$

and $\because \quad f(x,y) = 0$

$\therefore \quad y \to x$

$\therefore \quad u \to x, y \to x$

then $\quad \dfrac{du}{dx} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y}\cdot\dfrac{dy}{dx}$

$\because\ u = 0 \qquad \therefore\ \frac{du}{dx} = 0$

$$\therefore \qquad \frac{dy}{dx} = -\frac{\dfrac{\partial u}{\partial x}}{\dfrac{\partial u}{\partial y}}$$

$$\text{or,} \qquad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

If $u = f(x, y, z)$ where y and z are all functions of x, then we have

$u \to x, y, z \to x$ and

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx},$$

Also note that if $f(x, y, z) = 0$

then $$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx} = 0$$

(4) If $f(x, y) = 0$ then to find $\frac{d^2y}{dx^2}$,

We use the following notations :

$$p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}, r = \frac{\partial^2 f}{\partial x^2}, s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$$

$\because \qquad f(x, y) = 0$

$$\therefore \qquad \frac{d^2y}{dx^2} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\left(\frac{p}{q}\right) \qquad \text{(Result 5)}$$

$$\therefore \qquad \frac{d^2y}{dx^2} = -\left[\frac{q\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2}\right] \text{..................(i)}$$

$\because \qquad p, q \to x, y \to x$

$$\therefore \text{ and } \qquad \frac{dp}{dx} = \frac{\partial p}{\partial x} + \frac{\partial p}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\left(\frac{p}{q}\right)$$

$$= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial x}\left(\frac{p}{q}\right) = r - s\cdot\frac{p}{q} = \frac{rq - sp}{q}$$

$$\frac{dq}{dx} = \frac{\partial q}{\partial x} + \frac{\partial q}{\partial y}\cdot\frac{dy}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\left(\frac{p}{q}\right)$$

$$= \frac{\partial^2 f}{\partial x \cdot \partial y} - \frac{\partial^2 f}{\partial y^2}\cdot\frac{p}{q}$$

$$s - t\cdot\frac{p}{q}$$

$$= \frac{sq - pt}{q}$$

$$\therefore \text{from (i),} \quad \frac{d^2y}{dx^2} = -\frac{1}{q^2}\left[q\left\{\frac{rp - sp}{q}\right\} - p\left\{\frac{sq - pt}{q}\right\}\right] =$$

$$= -\frac{1}{q^3}\left[q^2 r - 2pqs + p^2 t\right]$$

SOME ADDITIONAL RESULTS

Partial Diffferentiation applied to :

(1) (1) Brackets: $\frac{\partial}{\partial x}[f(x,y,z)]^n = n[f(x,y,z)]^{n-1}\frac{\partial f}{\partial x}$

(2) Trignometric function: $\frac{\partial}{\partial x}\sin[f(x,y,z)] = \cos[f(x,y,z)]\cdot\frac{\partial f}{\partial x}$

(3) Expotential Function: $\frac{\partial}{\partial x}a^{[f(x,y,z)]} = a^{[f(x,y,z)]}\cdot\log a\cdot\frac{\partial f}{\partial x}$

(4) Log - function: $\frac{\partial}{\partial x}[\log\{f(x,y,z)\}] = \frac{1}{f(x,y,z)}\frac{\partial f}{\partial x}$

(5) Inverse Trignometric function:

$$\frac{\partial}{\partial x}\sin^{-1}[f(x,y,z)] = \frac{1}{\sqrt{1-f^2(x,y,z)}}\cdot\frac{\partial f}{\partial x}$$

Note : (1) In general, $\frac{\partial u}{\partial x} \neq \frac{1}{\frac{\partial x}{\partial u}}$

e.g. if $x = r\cos\theta$ and $y = r\sin\theta$

then $\left(\frac{\partial x}{\partial r}\right) = \cos\theta$

and since, $x^2 + y^2 = r^2$

$$2r\frac{\partial r}{\partial x} = 2x$$

$\therefore$ $\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta$

from (i) and (ii),

$$\frac{\partial x}{\partial r} \neq \frac{1}{\frac{\partial r}{\partial x}}$$

(2) When we write

$$u \to x, y, z$$

It means u depeds on x, y, z and x, y, z are independent among themselves.

EXAMPLES

TYPE - I

Note :

Problem in this type are based on direct differentiation

(1) First find dependent and independent variables.

(2) Use the necessary foumulae.

Examples 1 :

If $z = x^y + y^x$ then show that

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

sol. : $z = x^y + y^x$ $\therefore\ z \to x, y$

$$\frac{\partial z}{\partial y} = yx^{y-1} + y^x\cdot\log x \quad ....(i)$$

and $$\frac{\partial z}{\partial y} = x^y\log x + x\,y^{x-1} \quad ....(ii)$$

Differentiating (i) partially with respect to y,

$$\frac{\partial^2 z}{\partial x\,\partial y} = \left(y\,x^{y-1}\cdot\log x + x^{y-1}\right) + \left(y^x\cdot\frac{1}{y} + \log y\cdot xy^{x-1}\right)$$

$$= y\,x^{y-1}\cdot\log x + x^{y-1} + y^{x-1} + xy^{x-1}\cdot\log y..........(iii)$$

Differentiating (i) partially with respect to x,

$$\frac{\partial^2 z}{\partial x\,\partial y} = x^y\cdot\frac{1}{x} + yx^{y-1}\cdot\log x + 1\cdot y^{x-1} + xy^{x-1}\cdot\log y........(iv)$$

From (iii) and (iv)

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

Examples 2 :

If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ then

Show that : $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

Soln.: Note that u is a function of x, y, z

i.e. $u \to x, y, z$

$u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3x^2 - 3yz)$$

[see the rule of partial Differentiating applied to log function]

and similarly

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3y^2 - 3xz)$$

$$\frac{\partial u}{\partial x} = \frac{1}{(x^3 + y^3 + z^3 - 3xyz)}(3z^2 - 3xy)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} = \frac{3}{x+y+z}$$

Note that :

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} = \frac{-9}{(x+y+z)^2}$$

Examples 3 :

If $v = (1 - 2xy + y^2)^{-1/2}$ then show that

(i) $x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = y^2 v^3$ and (ii) $\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial v}{\partial y}\right\} = 0$

Sol.: (i) $v \to x, y$

We write $v^2 = 1 - 2xy + y^2$

Differentiate partially with respect to x and y,

(see the rule of Partial Differentiation applied to Brackets).

$$-2v^{-3}\frac{\partial v}{\partial x} = -2y$$

$$\therefore \quad \frac{\partial v}{\partial x} = v^3 y \qquad \text{......(1)}$$

and $-2v^3 \frac{\partial v}{\partial y} = -2x + 2y$

$$\frac{\partial v}{\partial y} = v^3(x - y) \qquad \text{..........(2)}$$

from 1 and 2

$$x\frac{\partial v}{\partial x} - y\frac{\partial v}{\partial y} = xy\,v^3 - yv^3(x - y) = y^2 v^3$$

(ii) $\because \quad (1 - x^2)\frac{\partial v}{\partial x} = (1 - x^2)yv^3$ from (1)

$$\therefore \quad \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial v}{\partial y}\right] = y\frac{\partial}{\partial x}\left[(1 - x^2)v^3\right]$$

($\because$ y is constant, and v is a function of x, y)

$$= y\left[-2xv^3 + (1 - x^2)3v^2\frac{\partial v}{\partial x}\right]$$

$$= y\left[-2xv^3 + 3(1 - x^2)v^2\, y\, v^3\right] \text{ from (1)}$$

$$= yv^3\left[-2x + 3(1 - x^2)yv^2\right] \qquad \text{.........(3)}$$

Again, $\frac{\partial}{\partial y}\left\{y^2 \frac{\partial v}{\partial y}\right\} = \frac{\partial}{\partial y}\left\{y^2 v^3 (x-y)\right\}$

$$= 3v^2 \frac{\partial v}{\partial y} x\left(x y^2 - y^3\right) + v^3 (2xy - 3y^2) \quad (\because v \to x, y)$$

$$= \left[3v^2 \cdot v^3 \cdot (x-y) \cdot (xy^2 - y^3) + v^3 (2xy - 3y^2)\right]$$

$$= v^3 y \left[3v^2 (x-y)(xy - y^2) + (2x - 3y)\right]$$

$$= v^3 y \left[3v^2 (x-y)(xy - y^2)(2x - 3y)\right] \quad ....(4)$$

$$\therefore \quad \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial y}\right\} + \frac{\partial}{\partial y}\left\{y^2 \frac{\partial v}{\partial y}\right\}$$

$$= y v^3 \left[-2x + 3y(1-x^2)v^2 + 3v^2 y (x - y^2) + (2x - 3y)\right] \text{ from (3), (4),}$$

$$= y v^3 \left[3y(1-x^2)(x-y^2)v^2 - 3y\right]$$

$$= y v^3 \left[3y\left(1 - x^2 + x^2 - 2xy + y^2\right)v^2 - 3y\right]$$

$$= y v^3 \left[3y\left[v^{-2}\right] v^2 - 3y\right]$$

$$= 0 \quad \because v^2 = 1 - 2xy + y^2$$

Example 4 :

If $u = \log(\tan x + \tan y + \tan z)$ then show that

$$\sin 2x \cdot \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2.$$

Soln.: Here $u \to x, y, z$

(Using the rule of partial diff. applied to log function) we have,

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 x$$

$$\frac{\partial u}{\partial y} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 y$$

$$\frac{\partial u}{\partial x} = \frac{1}{\tan x + \tan y + \tan z} \cdot \sec^2 z$$

$$\therefore \sin 2x \frac{\partial u}{\partial x} + \sin 2y \cdot \frac{\partial u}{\partial y} + \sin 2z \cdot \frac{\partial u}{\partial z}$$

$$= \frac{\sin 2x . \sec^2 x + \sin 2y . \sec^2 y + \sin 2z. \sec^2 z}{\tan x + \tan y + \tan z}$$

$$= \frac{2\left(\tan x + \tan y + \tan z\right)}{\left(\tan x + \tan y + \tan z\right)} = 2$$

$$\because \quad \sin 2x. \sec^2 x = 2 \sin x . \cos x . \frac{1}{\cos^2 x} = 2 \tan x$$

Examples 5 :

If $\theta = t^n, e^{-r^{2/4t}}$ then find the value of n so that $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}$

Sol.: We have $\theta \to r, t$

(To simplify the expression, we take log).

We have, $\quad \log \theta = n \log t - \frac{r^2}{rt}$

Diff. partially with respect to r,

$$\frac{1}{\theta}\frac{\partial \theta}{\partial r} = \frac{-2r}{4t}$$

$$\therefore \quad \frac{\partial \theta}{\partial r} = -\frac{r\theta}{2}$$

$$\therefore \quad r^2 \frac{\partial \theta}{\partial r} = -\frac{r^2 \theta}{2t}$$

Diff. partially with respect to r,

$$\therefore \quad \frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2}\left[3r^2 \theta + r^3 \frac{\partial \theta}{\partial r}\right] = -\frac{1}{2t}\left[3r^2 \theta - \frac{r^r \theta}{2t}\right] \text{ ...... from (1)}$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta}{\partial r}\right) = -\frac{1}{2}\left[3\theta - \frac{r^2 \theta}{2t}\right] \quad .... (2)$$

Again diff. given relation with respect to t partially ,

$$\frac{1}{\theta}\frac{\partial\theta}{\partial t} = \frac{n}{t} + \frac{r^2}{4}\cdot\frac{1}{t^2}$$

$$\therefore \quad \frac{\partial\theta}{\partial t} = \theta\left[\frac{n}{t} + \frac{r^2}{4t^2}\right]$$

$$\therefore \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial\theta}{\partial t},$$

From (2) and (3) we get,

$$\frac{n\theta}{t} + \frac{r^2\theta}{4t^2} = -\frac{3}{2}\frac{\theta}{t} + \frac{r^2\theta}{4t^2} \qquad \therefore\ n = -\frac{3}{2}$$

Example 6 :

If $u(x, t) = A e^{-gx}\cdot\sin(nt - gx)$

and if $\frac{\partial u}{\partial x} = u\frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\frac{n}{2u}}$

soln.: $u \to x, t$

We have, $\frac{\partial u}{\partial t} = Ae^{-gx}\cdot\cos(nt - gx)\cdot n$

$$= An\, e^{-gx}\cos(nt - gx)\ (\because x \text{ is to be kept constant}) ....(1)$$

Again diff. u partially with respect to x, we get,

$$\frac{\partial u}{\partial x} = A\left[-g\, e^{-gx}\cdot\sin(nt - gx) - g\cdot e^{-gx}.\cos(nt - gx)\right]$$

$$= -Ag\, e^{-gx}\left[\sin(nt - gx) + \cos(nt - gx)\right]$$

(Rule of partial differentitation applied to product)

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} = -Ag\left[-g\, e^{-gx}\sin(nt - gx) + \cos(nt - gx)\right]$$

$$e^{-gx}\left[-g\cdot\cos(nt - gx) + g\sin(nt - gx)\right]$$

$$= +Ag^2 e^{-gx}\left[2\cos(nt - gx)\right]$$

$$\because \quad \frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$$

$\therefore$ From (1) and (2)

$$(A\, n\, .e^{-gx}.\cos(nt - gx) = \mu.\,2.\,Ag^2.\,e^{gx}.\cos(nt - gx)$$

$$n = 2g^2\mu \qquad \therefore\ g = \sqrt{\frac{n}{2\mu}}$$

Example 7 :

If $u = x^2\tan^{-1}\frac{y}{x} - y^2\tan^{-1}\frac{x}{y}$, then find $\frac{\partial^2 u}{\partial x\,\partial y}$.

sol.: $u \to x, y$.

We have, $\frac{\partial u}{\partial y} = x^2\cdot\frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) - y^2\cdot\frac{1}{1+\frac{x^2}{y^2}}\left(-\frac{x}{y^2}\right) - 2y\tan^{-1}\frac{x}{y}$

$$= \frac{x^3}{x^2+y^2} - 2y\tan^{-1}\frac{x}{y} + \frac{xy^2}{x^2+y^2}$$

$$= \frac{x^3+xy^2}{x^2+y^2} - 2y\cdot\tan^{-1}\frac{x}{y}$$

$$= x - 2y\tan^{-1}\frac{x}{y}$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left[2 - 2y\tan^{-1}\frac{x}{y}\right]$$

$$= 1 - 2y\cdot\frac{1}{1+\frac{x^2}{y^2}}\left(\frac{1}{y}\right) = 1 - \frac{2y^2}{x^2+y^2} = \frac{x^2-y^2}{x^2+y^2}$$

Check Points :-

1) If $u,(x+y) = x^2+y^2$ then show that :

$$\left(\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right) = 4\left(1 - \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\left[\text{Hint } u = \frac{x^2+y^2}{x+y}\ \therefore u \to x, y \text{ find } \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} ....\right]$$

(2) Find the value of n so that $u = r^n (3\cos^2\theta - 1)$ satisfy the equation

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) = 0$$

$\left[\text{Hint} : u \to r, \theta, \text{Find } \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta} ....\right]$ Ans. n = 2, - 3.

(3) If $u = e^{xyz}$ then show that

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \left(1 + 3xyz + x^2 y^2 z^2\right) \cdot e^{xyz}$$

$$\left[\text{First find } \frac{\partial u}{\partial z} \text{ then } \frac{\partial^2 u}{\partial x\, \partial z} \text{ and } \frac{\partial^3 u}{\partial x \cdot \partial y\, \partial z}\right]$$

(4) If $v = \frac{c}{\sqrt{t}} e^{-\frac{x^2}{4a^2 t}}$, then prove that

$$\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$$

$\left[\text{Hint} : \text{Take log}, \therefore \log v = \log c - \frac{1}{2}\log t - \frac{x^2}{4a^2 t}\right]$ $v \to x, t$

(Find $\frac{\partial v}{\partial t}$ and $\frac{\partial^2 v}{\partial x^2}$, apply the rule of P.D. applied to log function)

(5) Find $\frac{\partial^2 u}{\partial y\, \partial z}$ where $u = \log(x^2 + y^2 + z^2)$ Ans. $\frac{-4yz}{(x^2 + y^2 + z^2)^2}$

(6) Verify $\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y}$

Where (i) $u = \log(y \sin x + x \sin y)$

(7) If $u = x^m y^n$ then show that

$$\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} = \frac{\partial^3 u}{\partial y\, \partial x^2}$$

(8) If $u = \log\left(y \sin x + x \sin x\right)$ then show that :

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial^2 u}{\partial y\, \partial y}$$

(9) If $u = \log\sqrt{x^2 + y^2 + z^2}$ then show that :

$$\left(x^2 + y^2 + z^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$$

[Hint : $\because$ $e^{2u} = x^2 + y^2 + z^2$ $\therefore u \to x, y, z$

$$\therefore\ 2e^{2u} \frac{\partial u}{\partial x} = 2x$$

$$\frac{\partial u}{\partial x} = x\, e^{-2u}$$

$$\frac{\partial^2 u}{\partial x^2} = e^{-2u} + x\left(-2e^{-2u}\right)\frac{\partial u}{\partial x} = e^{-2u} - 2xe^{-2e} \cdot x\, e^{-2e}$$

$$= e^{-2u} - 2x^2 e^{-4u}$$

10) If $u = r^m$, $r = \sqrt{x^2 + y^2 + z^2}$

Then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

$\left[\text{Hint} : u = \left(x^2 + y^2 + z^2\right)^{m/2} \quad u \to x, y, z \therefore \text{find } \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2} ....\right]$

Ans : $m(m + 1)r^m$

TYPE - II

Note :- Here we deal with the the problems of the type u = f (x, y, z) where x, y functions of x, y, z

i.e. $u \to X, Y, Z \quad \to x, y, z.$

We shall be frequently using the Chain- Rules can be develop drawing the flow- diagram.

Examples :

Example 1 :

Example 2 :

[Note that u is a function of only one variable $\frac{x^2}{y} = t$, which in turn is a function of x and y]

$\therefore \quad u \to t \to x, y \quad$ (see chain rule 2)

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{du}{dt} \cdot \frac{\partial t}{\partial x} = \frac{du}{dt} \cdot \frac{2x}{y}$$

$$\text{and,} \quad \frac{\partial u}{\partial y} = \frac{du}{dt} \cdot \frac{\partial t}{\partial y} = \frac{du}{dt} \cdot \left(\frac{-x^2}{y^2}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = \frac{2x^2}{y}\frac{du}{dt} - \frac{2x^2}{y}\frac{du}{dt} = 0$$

$$\text{i.e.} \; x\frac{\partial u}{\partial x} + 2y\frac{\partial u}{\partial y} = 0 \text{.......... ....(1)}$$

Diff (1) partially with repect to y we get,

$$x\frac{\partial^2 u}{\partial x^2} + 1 \cdot \frac{\partial u}{\partial x} + 2y\frac{\partial^2 u}{\partial x \partial y} = 0 \text{.......... ......(2)}$$

$$\text{and} \; x\frac{\partial^2 u}{\partial x \partial x} + 2y\frac{\partial^2 u}{\partial y^2} + 2 \cdot \frac{\partial u}{\partial y} = 0 \text{.......... ......(3)}$$

Taking (2) × x + (3) x y, we get

$$\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x\frac{\partial u}{\partial x} + xy\frac{\partial^2 u}{\partial x \partial y} + 2y\frac{\partial u}{\partial y} + 2y^2\frac{\partial^2 u}{\partial y^2}\right) = 0$$

$$\because \quad x\frac{\partial u}{\partial y} + 2y\frac{\partial u}{\partial y} = 0 \qquad \text{from (1)}$$

$$\therefore \; x^2\frac{\partial^2 u}{\partial x^2} + 3xy\frac{\partial^2 u}{\partial x \partial y} + 2y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Example 3 :

If $(\cos x)^y = (\sin y)^x$ then find $\frac{dy}{dx}$

soln. : Taking logs, we get,

$$y \log \cos x = x \log \sin y$$

Let $\quad f(x,y) = y \log \cos x - x\log \sin y = 0$

$\therefore \quad \frac{dy}{dx} = -\frac{df/dx}{df/dy}$

Now, $\quad \frac{\partial f}{\partial x} = y \cdot \frac{1}{\cos x}(-\sin x) - \log \sin y$

$\quad = -y \tan x - \log \sin y$

and $\quad \frac{\partial f}{\partial y} = \log \cos x - x\frac{1}{\sin y} \cos y = \log \cos x - x \cot y$

$\therefore$ From (1) $\frac{dy}{dx} = \frac{y \tan x + \log \sin y}{\log \cos x - x \cot y}$

Example 4:

If $y^x + x^y = (x+y)^{x+y}$ then find $\frac{dy}{dx}$

Soln.: Let $f(x+y) = (x+y)^{x+y} - y^x - x^y = o$

$\therefore \quad \frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$..................(1)

Now, $\frac{\partial f}{\partial x} = (x+y)^{(x+y)} \cdot [1+\log(x+y)] - y^x \cdot \log y - y\, x^{y-1}$

and $\quad \frac{\partial f}{\partial y} = (x+y)^{(x+y)} [1+\log(x+y)] - xy^{x-1}\, x^y \cdot \log x$

$\therefore$ From (1), $\frac{dy}{dx} = \frac{-\{(x+y)^{(x+y)} [1+\log(x+y)] - y^x \log y - yx^{y-1}\}}{\{(x+y)^{(x+y)} [1+\log(x+y)] - xy^{x-1} - x^y \cdot \log x\}}$

Example 5:

Pr ove that at a point of the surface

$x^x y^y z^z = c$

where $x = y = z$, $\frac{\partial^2 z}{\partial x\, \partial y} = -(x \log ex)^{-1}$

Sol : $\left(\text{From the expression } \frac{\partial^2 z}{\partial x\, \partial y} \text{ it is clear that } z \to x, y\right)$

Taking logs

$x \log x + y \log y + z \log z = \log c$

Different with respect to y partially, (i.e. keeping x constant)

$0 + \log y + y \cdot \frac{1}{y}\left(\log z + z \cdot \frac{1}{z}\right)\frac{\partial z}{\partial y} =$

$\frac{\partial^2 z}{\partial x} = -\frac{(1+\log y)}{(1+\log z)}$

Diff. with respect to x partially, we get,

$$\frac{\partial^2 z}{\partial x\, \partial y} = -(1+\log y)\left[\frac{-1}{(1+\log z)^2} \cdot \frac{1}{z}\frac{\partial z}{\partial x}\right]$$

$$= \frac{(1+\log y)}{z(1+\log z)^2}\frac{\partial z}{\partial x}$$

Now, we can show (as in (1) that $\frac{\partial z}{\partial x} = -\frac{(1+\log x)}{(1+\log z)}$

$\therefore \quad$ From (2), $\quad \frac{\partial^2 z}{dx\, dy} = -\frac{(1+\log x)(1+\log y)}{z(1+\log z)}$

At $\quad x = y = z$, $\quad \frac{\partial^2 z}{dx\, dy} = -\frac{(1+\log x)^2}{x(1+\log x)^3} = -\frac{1}{x(1+\log x)}$

$$= -\frac{1}{x(\log e + \log x)} = -\frac{1}{x \log(e x)}$$

$$= -[x \log(ex)]^{-1}$$

Check Points :-

(1) If $z = f(x, y, u, v)$ where u, v are functions of x, y then prove that

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$$

and write corresponding formulae for $\frac{\partial z}{\partial y}$

$$\left[\text{Hint}: z \to x, y, u, v \to x, y \quad \frac{\partial z}{\partial x} = \text{............}\right]$$

(2) If $v = f(x^2 + y^2 + z^2)$ then show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 4(x^2 + y^2 + z^2)\, f'(x^2 + y^2 + z^2) + 6f(x^2 + y^2 + z^2)$$

[Hint: Let $x^2 + y^2 + z^2 = u$

$\therefore \quad v \to u \to x, y, z$

$\therefore \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial v}\cdot\frac{\partial u}{\partial x}$ and proceed]

TYPE - III VARIABLE TO BE TREATED AS CONSTANT

Notations : $\left(\frac{\partial u}{\partial x}\right)$ means partial derivative of u with respect to x keeping Y constant.

To find $\left(\frac{\partial u}{\partial x}\right)_y$ we must have an equation in u, x and y only.

9.8 solved examples

If $x = r\cos\theta,\ y = r\sin\theta$ then show that

$$\left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = x^2 + y^2,$$ Where surffixes denote variables kept constant

Soln.: $\because x = r\cos\theta,\ y = r\sin\theta$

$$\left(\frac{\partial x}{\partial r}\right)_\theta = \cos_\theta, \left(\frac{\partial y}{\partial r}\right)_\theta = \sin\theta.$$

$$\therefore \left[x\left(\frac{\partial x}{\partial r}\right)_\theta + y\left(\frac{\partial y}{\partial r}\right)_\theta\right]^2 = [x\cos\theta + y\sin\theta]^2 = [r\cos^2\theta + r\sin^2\theta]^2$$

$$= r^2 = x^2 + y^2$$

Example 2

If $u = lx + my,\ v = mx - ly$, then show that :

(i) $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$ and

(ii) $\left(\frac{\partial y}{\partial v}\right)_x \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2}{l^2 + m^2}$

Sol. : We have

$u = lx + my$

$v = mx - ly$

(i) $\because \quad u = lx + my$

$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = l$ .... (1)

(ii) To find $\left(\frac{\partial x}{\partial u}\right)_v$ we must have relation between x, u and v

Eliminating y from the given relations, we get,

$$lu + mv = (l^2 + m^2)x$$

$$x = \frac{lu + mv}{l^2 + m^2}$$

$$\left(\frac{\partial x}{\partial u}\right)_v = \frac{1}{l^2 + m^2} \quad \text{.......... ..(2)}$$

From (1) and (2)

$$\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v = \frac{l^2}{l^2 + m^2}$$

(iii) $\because \quad v = mx - ly$ ....(3)

$\therefore$ Diff with respect to v keeping x constant,

$$l = 0 - l\left(\frac{\partial y}{\partial v}\right)_x$$

$$\therefore \quad \left(\frac{\partial y}{\partial y}\right)_x = \frac{1}{l}$$

(iv) To find $\left(\frac{\partial v}{\partial y}\right)_u$, we eliminate x from given relations,

i.e. $\quad mu - lv = (l^2 + m^2)y$

$$\therefore \quad 0 - l\left(\frac{\partial v}{\partial y}\right)_u = (l^2 + m^2)\cdot 1$$

$$\therefore \quad \left(\frac{\partial v}{\partial y}\right)_u = -\frac{l^2 + m^2}{l}$$

$\therefore$ From (4), (5)

$$\left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = \frac{l^2 + m^2}{l^2}$$

Example 3 :

If $f(x, y, z) = 0$ then show that

$$\left(\frac{\partial z}{\partial x}\right)_y = \frac{1}{\left(\frac{\partial x}{\partial z}\right)_y}$$

So ln: Here we use the result that if $f(x, y) = 0$

then $\quad \frac{\partial y}{\partial x}_y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$

(i) Here $f(x, y, z) = 0$. When y is kept constant,

We have, $\quad \left(\frac{\partial z}{\partial x}\right)_y = -\frac{\partial f/\partial x}{\partial f/\partial z}$

(ii) And $\quad \left(\frac{\partial x}{\partial z}\right)_y = -\frac{\partial f/\partial z}{\partial f/\partial x}$

From (1) and (2) $\left(\frac{\partial z}{\partial x}\right)_y = -\frac{1}{\left(\frac{\partial x}{\partial z}\right)_y}$

Example 4 :

If $x = \frac{\cos\theta}{u}, y = \frac{\sin\theta}{u}$, evaluate

$$\left(\frac{\partial x}{\partial u}\right)_\theta \cdot \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial u}{\partial x}\right)_\theta \cdot \left(\frac{\partial u}{\partial y}\right)_x$$

Sol. :

(i) $\because \quad x = \frac{\cos\theta}{u}$

$\therefore \quad \left(\frac{\partial x}{\partial u}\right)_\theta = -\frac{\cos\theta}{u^2}$ ....(1)

(ii) $\therefore \quad y = -\frac{\sin^2\theta}{u^2}$

$\therefore \quad \left(\frac{\partial y}{\partial u}\right)_\theta = \frac{\sin^2\theta}{u}$ ....(2)

(iii) To find $\left(\frac{\partial u}{\partial u}\right)_\theta$, we eliminate θ from given relations,

i.e. $\quad x^2 + y^2 = \frac{1}{u^2}$

or $\quad u^2 = \frac{1}{x^2+y^2}$ ....(A)

$\therefore \quad 2u\left(\frac{\partial u}{\partial x}\right)_y = \frac{-2x}{\left(x^2+y^2\right)^2}$

$\therefore \quad \left(\frac{\partial u}{\partial x}\right)_y = \frac{-x}{u\left(x^2+y^2\right)^2}$ ....(3)

(iv) and again $\quad u^2 = \frac{1}{x^2+y^2}$

$\therefore \quad 2u\left(\frac{\partial u}{\partial y}\right)_x = \frac{-2y}{\left(x^2+y^2\right)^2}$

$\therefore \quad \left(\frac{\partial u}{\partial y}\right)_y = \frac{-y}{u\left(x^2+y^2\right)^2}$ ....(4)

From (1), (2), (3), (4) we get,

$$\text{Required expression} = \left(-\frac{\cos\theta}{u^2}\right)\left(\frac{-x}{u\left(x^2+y^2\right)^2}\right) - \left(\frac{\sin\theta}{u^2}\right)\left(\frac{-y}{u\left(x^2+y^2\right)^2}\right)$$

$$= \frac{x\cos\theta + y\sin\theta}{u^3\left(x^2+y^2\right)^2}$$

but $\quad x\cos\theta + y\sin\theta = \frac{\cos^2\theta + \sin^2\theta}{u} = \frac{1}{u}$

$$\text{Re quired expression} = \frac{1}{u^4\left(x^2+y^2\right)^2}$$

$= \frac{1}{u^4} \cdot u^4 = 1 \quad$ (from A)

Example 5:

If $x + y + z + u + v = a$, $x^2 + y^2 + z^2 + u^2 + v^2 = b^2$,
where a, b are constants, prove that

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} \cdot \left(\frac{\partial x}{\partial u}\right)_{v,z} = \left(\frac{\partial v}{\partial y}\right)_{x,z} \cdot \left(\frac{\partial y}{\partial v}\right)_{u,z}$$

Sol,: To find $\left(\frac{\partial u}{\partial x}\right)_{y,z}$ we have to eliminate v from the given equations. And as this process will have to be repeated four times, we proceed in the following way.

Let $x + y + z + u + v = a$ ....(1)

$x^2 + y^2 + z^2 + u^2 + v^2 = b^2$..............(2)

differentiating with respect to x partially keeping y, z as constants, we get,

$$1 + \left(\frac{\partial u}{\partial x}\right)_{y,z} + \left(\frac{\partial v}{\partial x}\right)_{y,z} = 0 \text{ ....(3)}$$

and $\quad 2x + 2u\left(\frac{\partial u}{\partial x}\right)_{y,z} + 2y\left(\frac{\partial v}{\partial x}\right)_{y,z} = 0$ ....(4)

Solving the equations (3), (4) for $\left(\frac{\partial u}{\partial x}\right)_{y,z}$ by Cramer' s Rule we get

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} = \frac{v - x}{u - v} \text{ ....(5)}$$

Similarly differentiating (1), (2) partially with respect to y keeping x, z as constants, we have

$$1 + \left(\frac{\partial u}{\partial y}\right)_{x,z} + \left(\frac{\partial v}{\partial y}\right)_{x,z} + 2v\left(\frac{\partial v}{\partial y}\right)_{x,z} = 0 \text{..............(7)}$$

$$ey + ex\left(\frac{\partial u}{\partial y}\right)_{x,z} + 2v\left(\frac{\partial v}{\partial y}\right)_{x,z} = 0$$

Solving (6), (7) for $\left(\frac{\partial v}{\partial y}\right)_{x,z}$ we get

$$\left(\frac{\partial v}{\partial y}\right)_{x,z} = \frac{y - u}{u - v} \text{ ....(8)}$$

Similarly differentiating (1), (2) partially with respect to u treating v, z as constant we get,

$$\left(\frac{\partial x}{\partial u}\right)_{u,z} + \left(\frac{\partial u}{\partial u}\right)_{v,z} + 1 = 0 \quad \ldots\ldots\ldots\ldots(9)$$

and $2x\left(\frac{\partial x}{\partial u}\right)_{v,z} + 2y\left(\frac{\partial y}{\partial u}\right)_{v,z} + 2u = 0 \ldots\ldots\ldots\ldots(10)$

solving (9), (10) for $\left(\frac{\partial x}{\partial u}\right)_{v,z}$ we get,

$$\left(\frac{\partial x}{\partial u}\right)_{v,z} = \frac{y-u}{x-y} \quad \ldots\ldots\ldots\ldots\ldots\ldots(11)$$

Similarly differentiating (1), (2) partially with respect to v where u, z, are kept constants, we get,

$$\left(\frac{\partial x}{\partial v}\right)_{u,z} + \left(\frac{\partial y}{\partial v}\right)_{u,z} + 1 = 0 \quad \ldots..(12)$$

$$2x\left(\frac{\partial x}{\partial v}\right)_{u,z} + 2y\left(\frac{\partial y}{\partial v}\right)_{u,z} + 2v = 0 \ldots\ldots\ldots(13)$$

Solving equations (12), (13) for $\left(\frac{\partial y}{\partial v}\right)_{u,z}$ we get,

$$\left(\frac{\partial y}{\partial v}\right)_{u,z} = \frac{v-x}{x-y} \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

From (5), (11) and from (8), (14) we get,

$$\left(\frac{\partial u}{\partial x}\right)_{y,z} \cdot \left(\frac{\partial x}{\partial u}\right)_{v,z} = \frac{v-x}{u-v} \cdot \frac{y-u}{x-y} = \left(\frac{\partial v}{\partial y}\right)_{x,z} \cdot \left(\frac{\partial y}{\partial v}\right)_{u,z}$$

Example 6:

If $u = x^2 + y^2$ and

$x = s + 3t$

$y = 2s - t$

Sol. : We have $u = x^2 + y^2$

$\therefore \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$

Now, $u \to x, y \to s, t$

$$\therefore \quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= (2x)\,(1) + (2y)\,(2)$$

$$= 2x + 4y$$

Now, $$\frac{\partial^2 u}{\partial s^2} = \frac{\partial}{\partial s}\left(\frac{\partial u}{\partial s}\right) = \frac{\partial}{\partial s}(2x + 4y)$$

$$= 2\frac{\partial x}{\partial s} + 4\frac{\partial y}{\partial s}$$

$$= 2 \times 1 + 4 \times 2 = 10$$

And $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} \ldots\ldots\ldots\ \ldots(i)$$

$$= 2x \times 3 + 2y\,(-1)$$

$$= 6x - 2y$$

And $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t}(6x - 2y)$$

$$= 6\frac{\partial x}{\partial t} - \frac{\partial y}{\partial t} x_2$$

$$= 6(3) - 2(-1) = 20 \quad \ldots\ldots\ldots\ \ldots\ldots\ldots\ \ldots(ii)$$

Check Points :-

(1) If $x = r\cos\theta,\ y = r\sin\theta$ then show that $\left(\frac{\partial y}{\partial r}\right)_{\theta}\left(\frac{\partial y}{\partial r}\right)_{\theta} = 1$

(2) If $\phi(x, y, z) = 0$ then show that $\left(\frac{\partial z}{\partial y}\right)_x \cdot \left(\frac{\partial x}{\partial z}\right)_y \cdot \left(\frac{\partial y}{\partial x}\right)_z = -1$

(3) If $u = ax + by,\quad v = bx - ay$, show that $\left(\frac{\partial u}{\partial x}\right)_y \cdot \left(\frac{\partial x}{\partial u}\right)_v \cdot \left(\frac{\partial y}{\partial v}\right)_x \cdot \left(\frac{\partial v}{\partial y}\right)_u = 1$

B. Sc. (IT) -Sem- I

Questions should be wirtten one below the other and in every front pge only.

Duration : **Three Hours**. Total Marks assigned to the paper **100** marks

Marks assigned to each quesrion should be stated against each question.

Instructions to the candidates, if any :-

N.B. :

1) Question No. 1 is compulsory.
2) Answer any four from remaining Q2 to Q7 . Questions.
3) All Questions carry equal marks.

Q. No. **Marks**

Q.1

a) Find n^{th} ns derivative (5)

$y = 1/(x^2 + a^2)$

b) Solve : (5)

$$(3e^x \tan y)dx + (1 - e^x) \cdot \operatorname{Sec}^2 y \cdot dy = 0$$

c) Solve : (5)

$$y/x \cdot dy/dx = \sqrt{1 + x^2 + y^2 + x^2y^2}$$

d) If (5)

$$u = x^2 \tan^{-1} y/x - y^2 \tan^{-1} x/y, \text{then}$$
$$\text{find } \partial^2 u/\partial x \cdot \partial y$$

Q.2

a) Find the inverse of the matrix by finding its adjoint, Where, (7)

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$$

b) Find the inverse of the matrix using elementry transformation (7)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ 3 & 3 & 4 \end{pmatrix}$$

c) Examine for consistency : (6)

$$3x + y + 2z = 3$$
$$2x - 3y - z = -3$$
$$x + 2y + z = 4$$

Q.3

a) Examine for linear dependence : (7)

b) Find eigen values and eigen vectors for (7)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$$

c) Find eigen values and eigen vectors for (6)

$$A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$$

Q.4

a) Find the curve which passes through the points (2,1) and (8,2) for which subtangent at ant varies as the abscissa of that point. (7)

b) solve (7)

$$dy/dx = x(2\log x + 1)(\sin y + y\cos y)$$

c) From the differential equation if (6)

$$y = c_1 \cos x + c_2 \sin x$$

Q.5

a) solve : (7)

$$(x - y)^2 \cdot dy/dx = a^2$$

$$dy/dx = \cos(x + y) \quad (7)$$

c) A body of mass 'm' falling from rest is subject to the force of gravity and an air resistance proportional to the square of the velocity (i.e.kv2) . If it falls thorugh a distance x and possess a velocity v at that instant, show that

$$2kx/m = \log\left[a^2/\left(a^2 - v^2\right)\right] \text{ where } mg = ka^2$$ (6)

Q.6.

a) Solve (7)

$$\left(D^2 + 3D + 2\right) = \sin\left(e^x\right)$$

b) Solve : (7)

$$\left(D^2 + 3D + 2\right)y = e^{e^x}$$

c) Solve : (6)

$$\left(D^2 - 5D^2 + 8D - 4\right)y = e^{2x} + e^{2x} + 3e^{-x} + 2$$

Q.7

a) If $y^{1/m} + y^{-1/m} = 2x$ show that (7)

$$\left(x^2 - 1\right)y_{n+2} + (2n+1)\,xyn_{+1} + \left(n^2 - m^2\right)yn = 0$$

b) obtain the differential equation for the relation (7)

c) If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$ (6)

$$\left(\partial/\partial x + \partial/\partial y + \partial/\partial z\right)^2 u = -9\left(x + y + z\right)^2$$

Content

INSTITUTE OF DISTANCE EDUCATION

UNIVERSITY OF MUMBAI

B.SC (IT) & B.SC. (COMPUTER SCIENCE)

MATHEMATICS - I

Syllabus

B.Sc (IT) & Computer Science

MATHEMATICS - I

Matrices . Adjoint of a matrix, Inverse of matrix, Solving homogeneous & nonhomogeneous equations, Matrices . Linear dependence and independence of rows and column matrix, derogetory and non-derogetory matrices, Eigne values and Eigen vectors, Differential Equation of 1sr order 1st degree & application, Differential Equation of higher order & application, Successive differentiation, Mean value theorems, Partial Differentiation, Euler's theorem, Extreme values of function of two variables . application.

REFERENCE :

P. N. Wartikar & J. N. Wartikar, .Elements of Applied Mathematics., 7th, Pune Vidyarthi Graha,1988.

B. S. Grewal, .Higher Engineering Mathematics. Shanti Narayan, .Differential Calculus., Shamalal Charitable Trust, 1997.

Murray Spiegal, .Vector Analysis., McGraw Hill, 1974

TERM WORK :

Should contain at least 10 assignments covering the syllabus

TUTORIAL :

Tutorial should contain 5 assignments

Practical :

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B.SC (IT) & B.SC (COMPUTER SCIENCE)

MATHEMATICS - I