Practical 3.1 : Examples of Metric Spaces, Normed Linear Spaces.
Objective Questions 3.1

(1) Consider the following maps $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$.
(i) $d(x, y) = |x - 2y|$ (ii) $d(x, y) = |x^2 - y^2|$
(iii) $d(x, y) = |x - y|^2$ (iv) $d(x, y) = |x - y|^{\frac{1}{2}}$
(a) (iii) and (iv) are metrics on $\mathbb{R}$
(b) Only (iv) is a metric on $\mathbb{R}$
(c) (ii) and (iii) are metrics on $\mathbb{R}$
(d) (ii), (iii) and (iv) are metrics on $\mathbb{R}$

(2) Let d_1 and d_2 be metrics on a non-empty set X. Then
(a) $d_1^2 + d_2^2, ad_1$ where $a > 0$ are metrics on X, where $(d_1^2 + d_2^2)(x, y) = \Big(d_1(x, y)\Big)^2 + \Big(d_2(x, y)\Big)^2$ and $(ad_1)(x, y) = a\Big(d_1(x, y)\Big)$
(b) $\sqrt{d_1} + \sqrt{d_2}, ad_1$ where $a > 0$ are metrics on X, where $(\sqrt{d_1} + \sqrt{d_2})(x, y) = \sqrt{d_1(x, y)} + \sqrt{d_2(x, y)}$ and $(ad_1)(x, y) = a\Big(d_1(x, y)\Big)$
(c) $ad_1 + bd_2$ where $a, b \in \mathbb{R}$ is a metric on X, where $(ad_1 + bd_2)(x, y) = ad_1(x, y) + bd_2(x, y)$
(d) None of the above

(3) Consider the discrete metric d_1 defined on a non-empty set X by $d_1(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$.
Then for $x, y, z \in X$,
(a) $d_1(x, z) < d_1(x, y) + d_1(y, z)$
(b) $d_1(x, z) < d_1(x, y) + d_1(y, z)$ if and only if x, y, z are distinct.
(c) $d_1(x, z) = d_1(x, y) + d_1(y, z)$ if and only if $x = y = z$
(d) None of the above

(4) Let d_1 and d_2 be metrics on a non-empty set X. For $x, y \in X$, let $d(x, y) = \min\{d_1(x, y), d_2(x, y)\}$ and $d'(x, y) = \max\{d_1(x, y), d_2(x, y)\}$. Then
(a) Both d, d' are metrics on X. (b) d is a metirc on X, d' is not.
(c) d' is a metirc on X, d is not. (d) None of the above.

(5) Let (X, d_1) and (Y, d_2) be metric spaces. $d, d', d'' : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$ are defined as follows:
(i) $d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$
(ii) $d'((x_1, y_1), (x_2, y_2)) = [(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2]^{\frac{1}{2}}$
(iii) $d''((x_1, y_1), (x_2, y_2)) = [(d_1(x_1, x_2))^2 + (d_2(y_1, y_2))^2]$
(a) d, d', d'' are all metrics on $X \times Y$ (b) d, d' are metrics on $X \times Y$
(c) d', d'' are metrics on $X \times Y$ (d) None of the above.

(6) Let $(X, \| \ \|)$ be a normed linear space and $x, y, z \in X$. If d is the metric induced by the norm then

(a) $d(x+z, y+z) \geq d(x, y)$ and the strict inequality may hold.
(b) $d(x+z, y+z) \geq d(x, y) + d(y, z)$ and the strict inequality may hold.
(c) $d(x+z, y+z) = d(x, y)$.
(d) None of the above

(7) Consider the norms $\| \|_1, \| \|_2$ and $\| \|_\infty$ on $\mathbb{R}^2$, $\|x\|_1 = |x_1| + |x_2|$, $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$, $\|x\|_\infty = \max \{|x_1|, |x_2|\}$. Then
(a) $2\|x\|_\infty \leq \|x\|_2 \leq 2\|x\|_1$ (b) $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$
(c) $2\|x\|_\infty \leq \|x\|_1 \leq 2\|x\|_2$ (d) None of the above

(8) Let $X = C[0, 1]$ and consider the norms $\| \|_1, \| \|_\infty$ on X, where $\|f\|_1 = \int_0^1 |f(t)|\, dt$, $\|f\|_\infty = \sup \{|f(t)|, t \in [0, 1]\}$. Then for $f(t) = t, g(t) = t^2 \in X$, if d_1 and d_∞ are metric induced by $\| \|_1$, and $\| \|_\infty$ then
(a) $d_1(f, g) = \frac{1}{2}, d_\infty(f, g) = \frac{1}{3}$ (b) $d_1(f, g) = \frac{1}{6}, d_\infty(f, g) = \frac{1}{4}$
(c) $d_1(f, g) = \frac{1}{3}, d_\infty(f, g) = \frac{1}{2}$ (d) None of the above.

(9) Consider the normed linear space $(l^2, \| \|_2)$ where $l^2 = \{(x_n) : (x_n)$ is a sequence over $\mathbb{R}$, such that $\sum_{n=1}^{\infty} x_n^2 < \infty\}$ and for $x = (x_1, x_2, \ldots, x_n, \ldots)$, $\|x\|_2 = \sqrt{\sum_{n=1}^{\infty} x_n^2}$. Let $e_1 = (1, 0, 0, \ldots), e_2 = (0, 1, 0, 0, \ldots)$. Then for the metric d_2 induced by $\|\|_2$,
(a) $d_2(e_1 + e_2, e_1 - e_2) = \sqrt{2}$ (b) $d_2(e_1 + e_2, e_1 - e_2) = 2$
(c) $d_2(e_1 + e_2, e_1 - e_2) = \frac{1}{\sqrt{2}}$ (d) None of the above.

(10) Let X be the set of all real sequences $x = (x_n)$. Consider the metric d defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ \dfrac{1}{\min \{i : x_i \neq y_i\}} & \text{if } x \neq y \end{cases}$$

where $x = (x_n), y = (y_n) \in X$. Then for distinct sequences $x, y, z \in X$
(a) $d(x, z) \leq d(x, y) + d(y, z)$ and the equality may hold.
(b) $d(x, z) \leq \max \{d(x, y), d(y, z)\}$
(c) $d(x, z) \geq \max \{d(x, y), d(y, z)\}$
(d) None of the above.

(11) Let $(X, \| \|)$ be a normed linear space and d be the metric induced by $\| \|$. Then for $x, y, z \in X, d(x, z) = d(x, y) + d(y, z)$ if and only if
(a) $y = z$
(b) y lies on the segment joining x and z and between them.
(c) z lies on the segment joining x and y and between them.
(d) None of the above.

(12) Let X be a normed linear space and $x, y \in X$. Then
(a) $\|x-y\| \leq |\ \|x\| - \|y\|\ |$ (b) $\|x-y\| = |\ \|x\| - \|y\|\ |$
(c) $\|x-y\| \geq |\ \|x\| - \|y\|\ |$ (d) None of the above.

(13) Let $X = M_2(\mathbb{R})$. Consider the following maps from $X \longrightarrow \mathbb{R}$.
(i) $\|A\| = |\det A|$
(ii) $\|A\| = \sum_{1 \leq i,j \leq 2} |a_{ij}|$ where $A = (a_{ij})$
(iii) $\|A\| = \max_{1 \leq i,j \leq 2} |a_{ij}|$ where $A = (a_{ij})$
Then
(a) (i), (ii), (iii) are all norms on X.
(b) (ii) and (iii) are norms on X.
(c) (i) and (ii) are norms on X.
(d) None of the above.

Topology of Metric Spaces: Practical 3.1
Examples of Metric Spaces, Normed Linear Spaces
Descriptive Questions 3.1

(1) Let d_1 and d_2 be metrics on a non-empty set X. Check if the following are metrics on X. Justify your answer.

(i) d, where $d(x,y) = \max\{d_1(x,y), d_2(x,y)\}$ for $x, y \in X$
(ii) d, where $d(x,y) = \min\{d_1(x,y), d_2(x,y)\}$ for $x, y \in X$
(iii) d, where $d(x,y) = 2d_1(x,y) + 3d_2(x,y)$ for $x, y \in X$
(iv) d, where $d(x,y) = (d_1(x,y))^2 + (d_2(x,y))^2$ for $x, y \in X$
(v) d, where $d(x,y) = \max\{1, d_1(x,y), d_2(x,y)\}$ for $x, y \in X$

(2) Let (X, d) be a metric space. Show that the following are metrics on X.

(i) d_1 where $d(x,y) = \sqrt{d(x,y)}$
(ii) d, where $d(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$

(3) Show that d is a metric on $\mathbb{R}$, where $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ |x| + |y| & \text{if } x \neq y, x, y \in \mathbb{R} \end{cases}$

(4) Let $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R} \text{ for } 1 \leq i \leq n\}$. Show that $\|\ \|_1, \|\ \|_2$, and $\|\ \|_\infty$ are norms on $\mathbb{R}^n$ where for $x = (x_1, x_2, \ldots, x_n)$, $\|x\|_1 = \sum_{i=1}^{i=n} |x_i|$, $\|x\|_2 = \sqrt{\sum_{i=1}^{i=n} x_i^2}$ and $\|x\|_\infty = \max\{|x_1| : 1 \leq i \leq n\}$. Further, show that $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2 \leq n\|x\|_\infty$ for $x \in \mathbb{R}^n$

(5) Let $l^2 = \{(x_n) : (x_n)$ is a sequence of real numbers such that $\sum_{n=1}^{\infty} x_n^2 < \infty\}$. If $\|x\|_2 = \left(\sum_{n=1}^{\infty} x_n^2\right)^{\frac{1}{2}}$ for $x = (x_1, x_2, \ldots, x_n, \ldots) \in l^2$, then show theat $(l^2, \| \|_2)$ is a normed linear space.

(6) Let $X = C[0, 1]$ and show that $\| \|_1 : X \longrightarrow \mathbb{R}$ and $\| \|_\infty : X \longrightarrow \mathbb{R}$ defined by,
$\|f\|_1 = \int_0^1 |f(t)|\, dt$, $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$ are norms on X

(7) Let $X = C[0, 1]$ and consider the norms $\| \|_1$ and $\| \|_\infty$ defined by,
$\|f\|_1 = \int_0^1 |f(t)|\, dt$, $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$
Then for $f = t, g = t^2, h = t^3, t \in [0, 1]$, find $d_1(f, g), d_\infty(f, g), d_1(f, h), d_\infty(f, h)$ where d_1 and d_∞ are metrics induced by the norms $\| \|_1$ and $\| \|_\infty$ respectively.

(8) Let X be the set of real sequences

(i) Show that $d : X \times X \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned} d(x, y) &= 0 && \text{if } x = y \\ &= \frac{1}{\min\{i : x_i \neq y_i\}} && \text{if } x \neq y \end{aligned}$$

where $x = (x_n), y = (y_n) \in X$ is a metric on X.

(ii) Show that $d : X \times X \longrightarrow \mathbb{R}$ defined by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1 + |x_i - y_i|)}$$

where $x = (x_n), y = (y_n) \in X$ is a metric on X.

(iii) Let $X = \{(x_n) : (x_n)$ is a sequence of real numbers, $x_n \longrightarrow 0\}$. Show that $\| \| : X \longrightarrow \mathbb{R}$ defined by $\|x\| = \sup\{|x_n| : n \in \mathbb{N}\}$ for $x = (x_n)$ is a norm on X.

(9) Let $\| \|_2$ be the Euclidean norm on $\mathbb{R}^2$. Let $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} d(x, y) &= \|x\|_2 + \|y\|_2 && \text{if } x \neq y \\ &= 0 && \text{if } x = y \end{aligned}$$

for $x, y \in \mathbb{R}^2$. Show that d is a metric on $\mathbb{R}^2$

(10) Show that d is a metric on $\mathbb{N}$ where for $m, n \in \mathbb{N}$,

$$\begin{aligned} d(m, n) &= 0 && \text{if } m = n \\ &= 1 + \frac{1}{m + n} && \text{if } m \neq n \end{aligned}$$

(11) Show that $\| \ \|$ is a norm on X, where $X = M_2(\mathbb{R})$ and $\|A\| = \max_{1 \le i,j \le 2} |a_{ij}|$ for $A = (a_{ij})$

(12) Show that $\| \ \|_1$ is a norm on l^1 where $l^1 = \left\{ (x_n) : x_n \in \mathbb{R}, \sum_{n=1}^{\infty} |x_n| < \infty \right\}$ and $\|x\|_1 = \sum_{n=1}^{\infty} |x_n|$ for $x = (x_n)$

(13) Show that $\mathbb{C}$ (set of complex numbers) is a normed linear space where norm is the absolute value of a complex number.

Topology of Metric Spaces Practical 3.2
Sketching of Open Balls in $\mathbb{R}^2$, Open and Closed sets, Equivalent metric spaces
Objective Questions 3.2

. (Revised Syllabus 2018-19)

(1) In a metric space (X, d)
(a) an arbitrary intersection of open sets is an open sets.
(b) an arbitrary intersection of open balls is an open ball.
(c) an intersection of finitely many open balls is an open ball.
(d) None of the above.

(2) Let (X, d) be a metric space and $x, y \in X, r, s > 0$. If $B(x, r) = B(y, s)$, then
(a) $x = y$ and $r = s$ (b) $x = y$ but r may not be equal to s
(c) $r = s$ (d) None of the above

(3) Let (X, d) be a metric space and $x, y \in X, 0 < r < s$. Then
(a) $B(x, r) \subseteq B(x, s)$ and the equality may occur. (b) $B(x, r) \subsetneq B(x, s)$,
(c) $B(x, r) = B(x, s)$ if $r \geq 1$ (d) None of the above.

(4) Let (X, d) be a metric space in which the only open subsets are $\emptyset$ and X. Then
(a) d is a discrete metric on X.
(b) For $x, y \in X, d(x, y) \geq 1$ if $x \neq y$
(c) X is a singleton set.
(d) None of the above.

(5) Let G be a non-empty bounded open set in $\mathbb{R}^2$ with Euclidean metric. Then G is of the type
(a) $(a, b) \times (c, d)$, where $a, b, c, d \in \mathbb{R}, a < b, c < d$.
(b) $I \times J$, where I and J are union of finitely many bounded open intervals in $\mathbb{R}$
(c) $G_1 \times G_2$, where G_1 and G_2 are bounded open subsets of $\mathbb{R}$.
(d) None of the above.

(6) Consider the normed linear space $(\mathbb{R}^2, \| \|_1)$ where for $x = (x_1, x_2) \in \mathbb{R}^2, \|x\|_1 = |x_1| + |x_2|$. If $B_1((0, 0), 1)$ is an open ball with center $(0, 0)$ and radius 1, then
(a) $B_1((0, 0), 1)$ is a square with sides of length $\sqrt{2}$ which are parallel to coordinate axes.
(b) $B_1((0, 0), 1)$ is a square with sides of length $\sqrt{2}$ and diagonals are parallel to coordinate axes.
(c) $B_1((0, 0), 1)$ is a square with sides of length 2 which are parallel to coordinate axes.
(d) None of the above.

(7) Let (X, d) be a metric space and $x, y \in X$. Let $d(x, y) = s > 0$. Then $B(x, r) \cap B(y, r) = \emptyset$, if
(a) $r \geq \frac{s}{2}$ (b) $0 < r \leq \frac{s}{2}$ (c) $r \geq 2s$ (d) None of the above

(8) Consider the normed linear spaces $(\mathbb{R}^2, \| \|_1), (\mathbb{R}^2, \| \|_2)$ and $(\mathbb{R}^2, \| \|_\infty)$ where for $x = (x_1, x_2) \in \mathbb{R}^2$ $\|x\|_1 = |x_1| + |x_2|, \|x\|_2 = \sqrt{x_1^2 + x_2^2}, \|x\|_\infty =$ max $\{|x_1|, |x_2|\}$
If $B_1((0,0),1), B_2((0,0),1)$ and $B_\infty((0,0),1)$ denote open balls in $(\mathbb{R}^2, \| \|_1), (\mathbb{R}^2, \| \|_2)$ and $(\mathbb{R}^2, \| \|_\infty)$ respectively. Then
(a) $B_1((0,0),1) \subsetneq B_2((0,0),1) \subsetneq B_\infty((0,0),1)$
(b) $B_1((0,0),1) = B_2((0,0),1) = B_\infty((0,0),1)$
(c) $B_\infty((0,0),1) \subsetneq B_2((0,0),1) \subsetneq B_1((0,0),1)$
(d) None of the above.

(9) Let (X,d) be a metric space aned d_1 be the metric on X defined by $d_1(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ for $x, y \in X$
(a) Every open ball in (X, d_1) is an open ball in (X, d) and viceversa.
(b) Every open ball in (X, d_1) except possibly $B(x,r), r \geq 1$ for any $x \in X$ is an open ball in (X, d) .
(c) Every open ball in (X, d_1) is an open ball in (X, d)
(d) None of the above.

(10) Let $(X, \| \|)$ be a normed linear space. Let $A \subseteq X$ and U be an open subset of X in (X, d) where d is the metric induced by $\| \|$. Then
(a) $A + U$ is open if and only if A is open.
(b) $A + U$ is open.
(c) $A + U$ is open if and only if $A = \emptyset$ or A is a singleton set.
(d) None of the above.

(11) Let (X, d) be a metric space , $a \in X$ and $r' > r > 0$. Let $U_1 = \{x \in X : d(x,a) > r\}$, $U_2 = \{x \in X : d(x,a) \neq r\}$ and $U_3 = \{x \in X : r < d(x,a) < r'\}$. Then
(a) U_1 and U_2 are open subsets of X, but U_3 may not be open.
(b) U_1, U_2, U_3 are all open.
(c) U_1 is open subset of X, but U_2 and U_3 may not be open.
(d) None of the above.

(12) Consider the metric spaces $(\mathbb{N}, d)$ and $(\mathbb{N}, d_1)$ where d is the usual distance (induced from $\mathbb{R}$) and d_1 is the discrete metric in $\mathbb{N}$. Then
(a) d and d_1 are equivalent metrics on $\mathbb{N}$, but the two metric spaces do not have same open balls.
(b) The open balls in two metric spaces are the same.
(c) Every open ball in $(\mathbb{N}, d)$ is an open ball in $(\mathbb{N}, d_1)$
(d) None of the above.

(13) Consider the following subsets of $\mathbb{C}$ with respect to the usual distance

(i) $A = \{z \in \mathbb{C} : z = 2\} \bigcup \{z \in \mathbb{C} : |z| < 2\}$

(ii) $B = \{z \in \mathbb{C} : |\text{Re } z| < a\}$ where $a > 0, a \in \mathbb{R}$

(iii) $C = \{z \in \mathbb{C} : z \neq \dfrac{i}{n}, n \in \mathbb{N}\}$

(a) A, B and C are open.
(b) B, C are open.
(c) Only B is open.
(d) Only C is open.

(14) Consider the following subsets $(\mathbb{R}^3, d)$ where d Euclidean.
$E = \{(x, y, 0) \in \mathbb{R}^3\}$
$F = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = d, \text{ at least one of a, b, c is not zero}\}$
$G = \{(x, y, z) \in \mathbb{R}^3 : xyz \neq 0\}$. Then
(a) E, F and G are not open.
(b) Only G is open.
(c) F, G are open.
(d) Only E is open.

(15) Let $X = C[0, 1]$ with norm $\| \|_\infty$.
Let $E = \{f \in X : f(0) \neq 0\}$, $F = \{f \in X : f(\frac{1}{2}) \neq 0\}$. Then
(a) E is not open and F is open.
(b) Neither E nor F are open.
(c) Both E and F are open.
(d) E is open but F is not.

(16) Let $X = C[0, 1]$. Then
(a) $B_1(0, 1)$ is open in $(X, \| \|_\infty)$
(b) $B_1(0, 1) \subseteq B_\infty(0, r)$ for some $r > 0$.
(c) $B_\infty(0, 1) \subseteq B_1(0, r)$ for some $r > 0$.
(d) None of the above.

Topology of Metric Spaces: Practical 3.2
Sketching of Open Balls in $\mathbb{R}^2$, Open and Closed sets, Equivalent metric spaces
Descriptive Questions 3.2

(1) Give an example of a metric space in which $B(x, r) = B(y, s)$ but $x \neq y$ and $r \neq s$.

(2) Determine which of the following sets are open in the given metric space. Justify your answer in each case.

(i) $U = \{(x, y) \in \mathbb{R}^2 : xy \neq 0\}$ with Euclidean metric.

(ii) $U = \{(x, y) \in \mathbb{R}^2 : x = 0\}$ with Euclidean metric.

(iii) $\mathbb{Q}$ in $\mathbb{R}$ with usual distance.

(iv) $U = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 \leq 1\}$ with Euclidean metric.

(v) $U = \{(x, y) \in \mathbb{R}^2 : 2x + 3y < 1\}$ with Euclidean metric.

(ii) $U = B((0, 0), 1) \setminus \{(\frac{1}{2}, \frac{1}{2}), (\frac{1}{3}, \frac{1}{3})\} \in \mathbb{R}^2$ with Euclidean metric.

(3) Let (X, d) be a discrete metric space and $x \in X$. Find
(i) $B(x, \frac{1}{2})$ (ii) $B(x, \frac{3}{4})$ (iii) $B(x, 1)$ (iv) $B(x, r), r > 1$

(4) Draw open ball $B((0, 0), 1)$ in $\mathbb{R}^2$ with respect to the given metric.

(i) d_1 induced by the norm $\| \|_1, \|x\|_1 = |x_1| + |x_2|$ for $x = (x_1, x_2) \in \mathbb{R}^2$

(ii) d_2, the Euclidean metric.

(iii) d_1 induced by the norm $\| \|_\infty, \|x\|_\infty = \max \{|x_1|, |x_2|\}$ for $x = (x_1, x_2) \in \mathbb{R}^2$

(iv) d where $d(x, y) = 2|x_1 - y_1| + 3|x_2 - y_2|$ for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$

(5) Show that in the following examples U is open subset of $(\mathbb{R}^2, d)$, where d is the Euclidean metric. Also, for $p \in U$, find maximum r_p such that $B(p, r_p) \subseteq U$.

(i) $U = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}$.

(ii) $U = \{(x, y) \in \mathbb{R}^2 : x \notin \mathbb{Z}, y \notin \mathbb{Z}\}$.

(iii) $U = (0, 1) \times (0, 1)$.

(iv) $U = \{(x, y) \in \mathbb{R}^2 : -1 < x + y < 1\}$.

(6) Let $f, g \in C[0, 1]$ and suppose $f(t) < g(t)$ for each $t \in [0, 1]$. Show that $U = \{h \in C[0, 1] : f(t) < h(t) < g(t)$ for each $t \in [0, 1]\}$ is an open subset of $X = C[0, 1]$ under $\| \|_\infty$ norm where $\|f\|_\infty = \sup \{|f(t)| : t \in [0, 1]\}$

(7) Consider $X = C[0, 1]$ under the norms $\| \|_1$ and $\| \|_\infty$ where $\|f\|_1 = \int_0^1 |f(t)|\, dt$ and $\|f\|_\infty = \sup \{|f(t)| : t \in [0, 1]\}$. Draw the open ball $B(0, 1)$ in $(X, \| \|_1)$ and $(X, \| \|_\infty)$. (meaning show when does $f \in C[0, 1]$ lie in the open ball $B(0, 1)$).

(8) Describe the open balls $B(p, r)$ for $p \in \mathbb{Z}, r > 0$ considering cases $0 < r < 1, r = 1, r > 1$ in the subspace $\mathbb{Z}$ of $\mathbb{R}$ with usual distance.

(9) Let (X, d_1) and (Y, d_2) be metric spaces. Consider the metric $d : (X \times Y) \times (X \times Y) \longrightarrow \mathbb{R}$ defined by $d((x_1, y_1), (x_2, y_2)) = \max \{d_1(x_1, x_2), d_2(y_1, y_2)\}$. Let $p \in X, q \in Y$ and $r, s > 0$. Show that $B(p, r) \times B(q, s)$ is an open set in $(X \times Y, d)$.

(10) Consider the metric δ on $\mathbb{R}^2$ defined by

$$\begin{aligned} \delta(x, y) &= \|x\| + \|y\| \quad \text{if } x \neq y \\ &= 0 \quad \text{if } x = y \end{aligned}$$

for $x, y \in \mathbb{R}^2$ where $\| \|$ is the Euclidean norm in $\mathbb{R}^2$. Find the open balls $B((0, 0), r)$ and $B(x, r)$ where $x \neq (0, 0), \|x\| = \epsilon$ and $0 < \epsilon < r$

(11) Check whether the following subsets of $\mathbb{C}$ with respect to usual distance are open. Justify your answer.

1. $A = \{z \in \mathbb{C} : z = 2\} \bigcup \{z \in \mathbb{C} : |z| < 2\}$
2. $B = \{z \in \mathbb{C} : |\text{Re } z| < a, \text{ where } a \in \mathbb{R}^+\}$
3. $C = \{z \in \mathbb{C} : z \neq \frac{i}{n}, n \in \mathbb{N}\}$

(12) Let (X, d) be a metric space. We define a metric d' on $X \times X$ by

$$d'((x_1, x_2), (y_1, y_2)) = \max \{d(x_1, y_1), d(x_2, y_2)\}$$

Show that $D = \{(x, x) : x \in X\}$ is a closed subset of $(X \times X, d')$

(13) Show that $S' = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a closed subset of $(\mathbb{R}^2, \| \|_2), \| \|_2$ being the Euclidean metric.

(14) In the following examples, show that the given pairs of metrics are equivalent.

(i) For a metric space (X, d), the metrics d and d_1, where $d_1(x, y) = \dfrac{d(x, y)}{1 + d(x, y)}, x, y \in X$

(ii) For a metric space (X, d), the metrics d and d_1, where $d_1(x, y) = \min \{1, d(x, y)\}, x, y \in X$

(iii) On $\mathbb{N}, d$ and d_1 where d is the induced metric from the usual distance d in $\mathbb{R}$ and d_1 is the discrete metric.

(15) Let $X = C[0, 1]$ and d_1 and d_∞ be the metrics on X induced by $\| \|_1$ and $\| \|_\infty$. Prove or disprove d_1 and d_∞ are equivalent metrics on X.

(16) Let d_1, d_2, d_∞ be three metrics defined on $\mathbb{R}^2$ as follows:
$d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$, $\quad d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
$d_\infty(x, y) = \max \{|x_1 - y_1|, |x_2 - y_2|\}, \forall x = (x_1, x_2) \quad \& \quad y = (y_1, y_2)$.
Prove that d_1, d_2, d_∞ are equivalent metrics on $\mathbb{R}^2$ by showing

$$d_\infty(x, y) \leq d_2(x, y) \leq \sqrt{2} d_\infty(x, y) \text{ and } d_\infty(x, y) \leq d_1(x, y) \leq 2 d_\infty(x, y).$$

(17) Let d_1, d_2, d_∞ be three metrics defined on $\mathbb{R}^n$ as follows:
$d_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$, $\quad d_2(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$
$d_\infty(x, y) = \max \{|x_i - y_i| : 1 \leq i \leq n\}$
$\forall x = (x_1, x_2, \ldots, x_n) \quad \& \quad y = (y_1, y_2, \ldots, y_n)$
Show that $d_1(x, y) \geq d_2(x, y) \geq d_\infty(x, y) \geq n^{-\frac{1}{2}} d_2(x, y) \geq n^{-1} d_1(x, y)$

Topology of Metric Spaces: Practical 3.3
Subspaces, Interior points, Limit Points, Dense Sets and Separability, Diameter of a set, closure
Objective Questions 3.3

. (Revised Syllabus 2018-19)

(1) Consider the subspace $\mathbb{Z}$ of the metric subspace $\mathbb{R}$ with usual distance. Then
(a) Every open ball in $\mathbb{Z}$ is an infinite set.
(b) Every open ball in $\mathbb{Z}$ is a singleton set.
(c) Every open ball in $\mathbb{Z}$ is a finite set.
(d) None of the above.

(2) Let (X, d) be a metric space and $A, B \subseteq X$. Then
(a) $(A \cup B)^\circ = A^\circ \cup B^\circ, (A \cap B)^\circ = A^\circ \cap B^\circ$ (b) $(A \cup B)^\circ \subseteq A^\circ \cup B^\circ, (A \cap B)^\circ \subseteq A^\circ \cap B^\circ$
(c) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ, (A \cap B)^\circ = A^\circ \cap B^\circ$ (d) None of the above.

(3) Let A be a non-empty subset of $\mathbb{R}$, (distance being usual) then A° can be
(a) empty
(b) singleton set
(c) a finite set containing more than one element
(d) countable but not finite

(4) Consider $A = [0, 1)$ with the induced distance from the usual distance in $\mathbb{R}$. Then
(a) An open ball in A is of the type $(-r, r)$ with $0 < r < 1$
(b) $[0, \frac{1}{2})$ is an open ball in A
(c) $[0, 1)$ is not an open ball in A
(d) None of the above

(5) In the subspace $(\mathbb{Q}, d)$ of $(\mathbb{R}, d)$ where d is the usual distance in $\mathbb{R}$, $E = \{r \in \mathbb{Q} : 2 < r^2 < 3\}$ is
(a) an open ball (b) an open set which is not bounded.
(c) open and closed (d) None of the above.

(6) Let A be a closed subset of $\mathbb{R}$ (distance usual) $A \neq \emptyset, A \neq \mathbb{R}$. Then
(a) $A = \overline{(A^\circ)}$
(b) A is countable.
(c) A is not open.
(d) A is a bounded set.

(7) Let (X, d) be a metric space and $A, B \subseteq X$. Let $D(S)$ denote the set of limit points of $S \subseteq X$. Then
(a) If $A \subsetneq B$, then $D(A) \subsetneq D(B)$
(b) If $A \subsetneq B$, then $D(B) \subsetneq D(A)$
(c) If $A \subsetneq B$, then $D(A) \subseteq D(B)$ and the equality may occur.
(d) None of the above.

(8) Let d be the usual distance on $\mathbb{R}$ and d_1 be the discrete metric on $\mathbb{R}$. Let $A = (0, 1)$. If $D(A)$ denotes the set of all limit points of A, then
(a) In $(\mathbb{R}, d), D(A) = (0, 1)$ and in $(\mathbb{R}, d_1), D(A) = \{0, 1\}$
(b) In $(\mathbb{R}, d), D(A) = [0, 1]$ and in $(\mathbb{R}, d_1), D(A) = \emptyset$
(c) In $(\mathbb{R}, d), D(A) = (0, 1)$ and in $(\mathbb{R}, d_1), D(A) = (0, 1)$
(d) None of the above.

(9) Consider the following subsets of $\mathbb{R}$ (distance in $\mathbb{R}$ being usual):
(i) $\mathbb{N}$ (ii) $\mathbb{Q}$ (iii) $\{\frac{1}{n} : n \in \mathbb{N}\}$ (iv) $(-1, 0)$. Then 0 is a limit point of
(a) (iv) only
(b) (ii), (iii) and (iv)
(c) (ii) and (iv) only
(d) $\mathbb{N}$

(10) Let (X, d) be a metric space and $A, B \subseteq X$. Then
(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}, \overline{A \cap B} = \overline{A} \cap \overline{B}$
(b) $\overline{A \cup B} \subset \overline{A} \cup \overline{B}, \overline{A \cap B} = \overline{A} \cap \overline{B}$
(c) $\overline{A \cup B} = \overline{A} \cup \overline{B}, \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$
(d) None of the above

(11) Let (X, d) be a metric space and $A \subseteq X$. If $G \subseteq X$ is an open set such that $G \cap A = \emptyset$ then
(a) $\overline{G} \cap A = \emptyset$ (b) $G \cap \overline{A} = \emptyset$ (c) $\overline{G} \cap \overline{A} = \emptyset$ (d) None of the above

(12) Let $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\}$ in $\mathbb{R}$ where the distance is usual. Then
(a) A is a closed set. (b) A is not a closed set, $\overline{A} = (0, 1]$
(c) A is not a closed set, $\overline{A} = [0, 1]$
(d) None of the above.

(13) Consider $Y = [0, 1] \subseteq \mathbb{R}$, with the induced usual distance d of $\mathbb{R}$. Let $A = [0, 1) \subseteq Y$. Then in (Y, d)
(a) $\partial A = (0, 1)$ (b) $\partial A = \{0, 1\}$ (c) $\partial A = \{1\}$ (d) None of the above.

(14) Consider $\mathbb{N}$ with the induced usual distance of $\mathbb{R}$. Let $A = \{1, 2, \ldots, 10\} \subseteq \mathbb{N}$. Then the statement which is **not true** in $(\mathbb{N}, d)$ is
(a) $A^\circ = \emptyset$ (b) $\overline{A} = A$ (c) $\partial A = \emptyset$ (d) None of the above.

(15) Let $A, B \subset \mathbb{R}$, and d be the usual distance in $\mathbb{R}$. Then
(a) $d(A^\circ, B^\circ) = d(A, B) = d(\overline{A}, \overline{B})$ (b) $d(A, B) = d(\overline{A}, \overline{B})$
(c) $d(A^\circ, B^\circ) = d(A, B)$. (d) None of the above.

(16) Let (X, d) be a metric space and $A, B \subseteq X$ such that A, B are non-empty and $A \cap B = \emptyset$. Then
(a) $d(A, B) > 0$ (b) $d(A, B) > 0$ if A, B are open.
(c) $d(A, B) > 0$ if A, B are closed. (d) None of the above.

(17) Let $S^1 = \{(x, y) : x^2 + y^2 = 1\} \subseteq \mathbb{R}^2$, distance d being Euclidean. For $p \in \mathbb{R}^2, d(p, S^1)$ equals
(a) $\|p\|$ (b) $\|p\| - 1$ (c) $\|p\| + 1$ (d) None of the above.

(18) Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \cdots\}$ (distance in $\mathbb{R}$ usual). Then $\overline{A}$ equals
(a) $[0, 1]$ (b) $(0, 1)$ (c) $[0, 1] \cap \mathbb{Q}$ (d) $\{\frac{m}{2^n}, m, n \in \mathbb{N}\} \cap [0, 1]$

(19) Consider the set $A = \{1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \cdots\}$ (distance in $\mathbb{R}$ usual). Then $\overline{A}$ equals
(a) A is a closed set (b) A is not a closed set, $\overline{A} = (0, 1]$.
(c) A is not a closed set, $\overline{A} = [0, 1]$ (d) None of the above.

(20) Let $A = \left\{\dfrac{|x|}{1 + |x|} : x \in \mathbb{R}\right\}$, (distance usual). Then the set of all limit points of A is
(a) $(0, 1]$ (b) $(0, \infty)$ (c) $[0, 1]$ (d) None of the above.

(21) Let $A = \left\{\dfrac{x}{1 + |x|} : x \in \mathbb{R}\right\}$, (distance usual). Then the set of all limit points of A is
(a) $(-1, 1)$ (b) $[-1, 1]$ (c) $(0, \infty)$ (d) None of the above.

Topology of Metric Spaces: Practical 3.3
Subspaces, Interior points, Limit Points, Dense Sets and Separability, Diameter of a set, closure
Descriptive Questions 3.3

(1) Give an example of a metric space $(X, d), A, B \subseteq X$ such that $A^\circ = B^\circ = \emptyset$ but $(A \cup B)^\circ = X$

(2) Find the interiors of the following subsets in a given metric space.

(i) $\mathbb{Z}$ in $(\mathbb{R}, d)$ where d is the usual distance.
(ii) $\mathbb{Q}$ in $(\mathbb{R}, d)$ where d is the usual distance.
(iii) $\{(x, y) \in \mathbb{R}^2 : x > y\} \cup \{(0, 0)\}$ in $(\mathbb{R}^2, d)$ where d is the Euclidean metric.

(3) Find the closure of the following subsets of $\mathbb{C}$ (distance being usual)

(i) $S = \{z = \frac{i}{n} : n \in \mathbb{N}\}$
(ii) $S = \{z = \frac{1}{m} + \frac{i}{n} : m, n \in \mathbb{N}\}$
(iii) $S = \{z = x + iy, x, y \in (0, 1), x, y \in \mathbb{Q}\}$
(iv) $S = \{z = x + iy, x, y \in (0, 1)\}$

(4) Consider the subspace $A = [0, 1)$ of $\mathbb{R}$ where distance in $\mathbb{R}$ is usual. Find $B_A(0, r)$ an open ball in the subspace A for $r > 0$

(5) Consider the subspace $A = [0, \infty)$ of $\mathbb{R}$ where distance in $\mathbb{R}$ is usual. Find $B_A(0, 1)$ an open ball in the subspace (A, d).

(6) Show that $A = \{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\}$ is both open and closed in the subspace $\mathbb{Q}$ of $\mathbb{R}$ with usual distance.

(7) Prove or disprove : Let (X, d) be a metric space and $A \subseteq X$. Then
(i) $\overline{(A^\circ)} = \overline{A}$ (ii) $(\overline{A})^\circ = A^\circ$

(8) In $\mathbb{R}$, with respect to usual distance, show that $A = \mathbb{N}, B = \{n + \frac{1}{n} : n \in \mathbb{N}, n \neq 1\}$ are closed sets such that $A \cap B = \emptyset$. Also find $d(A, B)$.

(9) (i) In $(\mathbb{R}, d)$, where d is the usual distance, find $d(Q, \mathbb{R} \setminus \mathbb{Q})$ and $d(\mathbb{Q}, A)$ where A is any non-empty subset of $\mathbb{R}$.

(ii) In $(\mathbb{R}^2, d), d$ being Euclidean, find $d(A, B)$ where $A = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ and $B = \{(x, y) \in \mathbb{R}^2 : xy = 1\}$.

Topology of Metric Spaces: Practical 3.4
Limit Points, Sequences, Bounded, Convergent and Cauchy Sequences in a Metric Space
Objective Questions 3.4

. (Revised Syllabus 2018-19)

(1) Let (x_n) be a sequence in a metric space $(X,d), x_n \longrightarrow p$. Let $A = \{x_n : n \in \mathbb{N}\}$. Then
- (a) p is a limit point of A
- (b) $p \in \overline{A}$
- (c) There is a subsequence (x_{n_k}) of (x_n) having distinct terms such that $x_{n_k} \longrightarrow p$
- (d) None of the above.

(2) Let S be an infinite subset of $\mathbb{R}$ such that $S \cap \mathbb{Q} = \emptyset$. Then
- (a) S has a limit point which belongs to $\mathbb{R} \setminus \mathbb{Q}$.
- (b) S has a limit point which belongs to $\mathbb{Q}$.
- (c) S is not closed.
- (d) $\mathbb{R} \setminus S$ has a limit point which is in S.

(3) Let d_1 and d_2 be equivalent metrics on X and (x_n) be a sequence in X. Then
- (a) (x_n) is bounded in $(X,d_1) \Longleftrightarrow (x_n)$ is bounded in (X,d_2).
- (b) (x_n) is convergent in $(X,d_1) \Longleftrightarrow (x_n)$ is convergent in (X,d_2).
- (c) (x_n) is a Cauchy sequence in $(X,d_1) \Longleftrightarrow (x_n)$ is a Cauchy sequence in (X,d_2).
- (d) None of the above.

(4) Every Cauchy sequence is eventually constant in
- (a) $(\mathbb{N},d)$ where d is usual.
- (b) $(\mathbb{Q},d)$ where d is usual.
- (c) $(\mathbb{R} \setminus \mathbb{Q},d)$ where d is usual.
- (d) None of the above.

(5) d and d_1 are metrics on $X = (0,\infty)$ where d is the usual distance and $d_1(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Then
- (a) If (x_n) is a Cauchy sequence in (X,d_1) then (x_n) is a Cauchy sequence in (X,d)
- (b) If (x_n) is a Cauchy sequence in (X,d) then (x_n) is a Cauchy sequence in (X,d_1)
- (c) If (x_n) is Cauchy in $(X,d_1), (x_n)$ may not be Cauchy in (X,d).
- (d) (x_n) is a Cauchy sequence in $(X,d) \Longleftrightarrow (x_n)$ is Cauchy sequence in (X,d_1)

(6) d and d_1 are metrics on $X = (0,\infty)$ where d is the usual distance and $d_1(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$. Then
- (a) If (x_n) is a bounded sequence in (X,d_1) then (x_n) is a bounded sequence in (X,d)
- (b) If (x_n) is a bounded sequence in (X,d) then (x_n) is a bounded sequence in (X,d_1)
- (c) If (x_n) is bounded in $(X,d_1), (x_n)$ may not be bounded in (X,d).
- (d) (x_n) is a bounded sequence in $(X,d) \Longleftrightarrow (x_n)$ is bounded sequence in (X,d_1)

(7) Let d_1 and d_2 be metrics on X such that $k_1 d_2(x,y) \leq d_1(x,y) \leq k_2 d_2(x,y)$ for all $x, y \in X$ where $k_1, k_2 > 0$ are constants. The the statement which is not true is
(a) (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2).
(b) $x_n \longrightarrow p$ in (X, d_1) if and only if $x_n \longrightarrow p$ in (X, d_2).
(c) (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2).
(d) None of the above.

(8) Consider the sequence (x_k) defined by $x_k = \left((-1)^k, \dfrac{1}{k}\right)$ in $\mathbb{R}^2$. d and d_1 are metrics on $\mathbb{R}^2$ where d is the Euclidean distance and d_1 is discrete metric. Then
(a) (x_k) is not bounded in $(\mathbb{R}^2, d)$ and $(\mathbb{R}^2, d_1)$. (b) (x_k) converges in $(\mathbb{R}^2, d)$.
(c) (x_k) has a convergent subsequence in $(\mathbb{R}^2, d)$. (d) (x_k) converges in $(\mathbb{R}^2, d_1)$.

(9) Let $x_k \longrightarrow x$ and $y_k \longrightarrow y$ in $(\mathbb{R}^n, d)$, d is Euclidean distance. Which statement is not true?
(a) $\|x_k\| \longrightarrow \|x\|$ and $\|y_k\| \longrightarrow \|y\|$.
(b) $\langle x_k, y_k \rangle \longrightarrow \langle x, y \rangle$
(c) x is a limit point of the set $A = \{x_k : k \in \mathbb{N}\}$ and y is a limit point of the set $B = \{y_k : k \in \mathbb{N}\}$
(d) $x_k + y_k \longrightarrow x + y$

(10) Consider $X = C[0,1], \|f\|_1 = \displaystyle\int_0^1 |f(t)| \, dt, \|f\|_\infty = \sup \{|f(t)| : t \in [0,1]\} \quad \forall f \in X$ and $f_n(x) = x^n$. Then
(a) $\{f_n\}$ converges in $(X, \| \, \|_1)$ but not in $(X, \| \, \|_\infty)$
(b) $\{f_n\}$ converges in $(X, \| \, \|_\infty)$ but not in $(X, \| \, \|_1)$
(c) $\{f_n\}$ does not converge in both.
(d) $\{f_n\}$ converges in both.

(11) Consider $(\mathbb{N}, d)$ where $d(m,n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \dfrac{1}{m+n} & \text{if } m \neq n \end{cases}$ Then
(a) Every sequence in $(\mathbb{N}, d)$ is bounded.
(b) Every sequence in $(\mathbb{N}, d)$ is eventually constant.
(c) Every Cauchy sequence in $(\mathbb{N}, d)$ is eventually constant.
(d) Every sequence in $(\mathbb{N}, d)$ is Cauchy.

(12) Consider the sequence $x_n = n - [\sqrt{n}]$ in $(\mathbb{R}, d)$ where d is usual metric. Then
(a) (x_n) is Cauchy.
(b) (x_n) is monotone increasing.
(c) (x_n) is monotone decreasing.
(d) (x_n) is not convergent but has a convergent subsequence.

(13) Let d_1 and d_2 be two metrics on X and there exists real numbers $k_1, k_2 > 0$ such that $k_1 d_2(x,y) \leq d_1(x,y) \leq k_2 d_2(x,y) \quad \forall x, y \in X$. Mark the sentences which is not true.
(a) (x_n) is a Cauchy sequence in (X, d_1) implies (x_n) is a Cauchy sequence in (X, d_2)
(b) (x_n) is a bounded sequence in (X, d_1) implies (x_n) is a bounded sequence in (X, d_2)
(c) (x_n) is a convergent sequence in (X, d_1) implies (x_n) is a convergent sequence in (X, d_2)
(d) (a), (b) and (c) are not true.

(14) The sequence $\left(\frac{1}{n}\right)$ is **not** convergent in
(a) $[0,1]$ with usual distance.
(b) $[0,1]$ with discrete metric.
(c) $\mathbb{Q}$ with usual distance. (d) $[0, \infty)$ with usual distance.

(15) The Cauchy sequence which is convergent in $(\mathbb{Q}, d)$, where d is the usual distance, is
(a) (x_n), where $x_n = 1 + \frac{1}{1!} + + \frac{1}{2!} \cdots + + \frac{1}{n!}$
(b) (x_n) where $x_1 = 1$ and $x_n = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$
(c) $(x_n) = \{0.1, 0.101, 0.101001, 0.1010010001, \cdots\}$ (d) (x_n) where $x_n = \frac{1}{n}\left(1 + \frac{1}{n}\right)^n$

Topology of Metric Spaces: Practical 3.4
Sequences, convergent and Cauchy sequences in a metric space
Descriptive Questions 3.4

(1) Show that the following sequences in $\mathbb{R}^2$ are convergent, distance being Euclidean.

(i) (x_n) where $x_n = \left(\frac{1}{n^2}, \frac{n^2-1}{n^3+1}\right)$

(ii) (x_n), where $x_n = \left(2^n, \frac{1}{n}\right)$ for $n \leq 9$ and $x_n = \left(2^{10}, \frac{-1}{n}\right)$ for $n \geq 10$

(2) Prove or disprove: Let d_1, d_2 be equivalent metrics on a non-empty set X. Then

(i) (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2)

(ii) (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2)

(3) Let d_1 and d_2 be equivalent metrics on a non-empty set X such that there exist $k_1, k_2 > 0$ such that

$$k_1 d_1(x,y) \leq d_2(x,y) \leq d_2 d_1(x,y) \quad \forall x, y \in X$$

Then show that

(i) (x_n) is bounded in (X, d_1) if and only if (x_n) is bounded in (X, d_2)

(ii) (x_n) is Cauchy in (X, d_1) if and only if (x_n) is Cauchy in (X, d_2)

(4) Show that the sequence $x_n = \frac{1}{n}$ converges to 0 in the usual metric space $\mathbb{R}$ but is not convergent in $X = (0, 1)$ with the usual metric.

(5) $X = C[0, 1]$. Show that $f_n(t) = e^{-nt}$ converges to 0 w.r.t. the metric $d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$ but is not convergent w.r.t. the metric $d_\infty(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$

(6) Let (X, d) be a metric space. If (x_n) and (y_n) are sequences in X such that $x_n \longrightarrow x$ and $y_n \longrightarrow y$, then prove that the sequence $d(x_n, y_n) \longrightarrow d(x, y)$ in $\mathbb{R}$ w.r.t. the usual metric.

(7) Let $X = C[0, 1]$ be a metric space with the metric d_∞ defined by

$$d_\infty(f, g) = \sup\{|f(t) - g(t)| : t \in [0, 1]\}$$

Show that the sequence $\{f_n\}$ in X given by $f_n(t) = \dfrac{nt}{n+t} \quad \forall t \in [0, 1]$, is a Cauchy sequence in X.

(8) Prove that every Cauchy sequence in a discrete metric space is convergent.

(9) Let (x_n) be a Cauchy sequence in a metric space (X, d) and (x_{n_k}) be a subsequence of (x_n). Show that $d(x_n, x_{n_k}) \longrightarrow 0$ in $\mathbb{R}$ w.r.t. the usual metric.

(10) Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Prove that $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$ w.r.t. the usual distance.

(11) Let (X, d) be a metric space and d' be a metric on X defined by

$$d'(x, y) = \min\{1, d(x, y)\}$$

Show that (x_n) is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in (X, d').

(12) Let (X, d_1) be a metric space and (x_n) be a sequence in X. Show that $x_n \longrightarrow x$ in (X, d_1) if and only if $d_1(x_n, x) \longrightarrow 0$ in $(\mathbb{R}, d)$ where d is the usual distance in $\mathbb{R}$.

(13) Let (a_n) and (b_n) be sequences in a metric space (X, d_1) and $x_n = d(a_n, b_n)$.If (a_n) is a Cauchy sequence in (X, d_1) and $x_n \longrightarrow 0$ in $(\mathbb{R}, d)$ (d is the usual distance), then show that (b_n) is a Cauchy sequence.

Topology of Metric Spaces: Practical 3.5
Complete Metric Spaces
Objective questions 3.5

. (Revised Syllabus 2018-19)

(1) $F_n = [n, \infty)$ for each $n \in \mathbb{N}$. Then $\cap_{n\in\mathbb{N}} F_n$
(a) has infinitely many points (b) is a singleton set.
(c) is the empty set. (d) None of the above .

(2) In $\mathbb{R}$ with respect to usual distance $\cap_{n\in\mathbb{N}} F_n$ is a singleton set when
(a) $F_n = [-n, n]$ (b) $F_n = [n, n+1]$ (c) $F_n = [1 - \frac{1}{n}, 1]$ (d) $F_n = [0, n]$

(3) $\displaystyle\bigcap_{n\in\mathbb{N}} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ is
(a) $\{1\}$ (b) $(0, 2)$ (c) empty (d) None of these.

(4) $\displaystyle\bigcap_{n\in\mathbb{N}} (-n, n)$ is
(a) $[-1, 1]$ (b) $(-1, 1)$ (c) empty (d) None of these.

(5) $\displaystyle\bigcap_{n\in\mathbb{N}} \left[-\frac{1}{n}, \frac{1}{n}\right]$ is
(a) $\{0\}$ (b) $[-1, 1]$ (c) $[0, 1]$ (d) None of these.

(6) $\displaystyle\bigcap_{n\in\mathbb{N}} \left[0, \frac{1}{n}\right]$
(a) $\{0\}$ (b) empty (c) $[0, 1]$ (d) None of these.

(7) $f : \mathbb{R} \longrightarrow \mathbb{R}$ be any function (distance is usual). Then
(a) f is continuous on $\mathbb{R}$ if and only if f satisfies intermediate value property.
(b) If f is continuous on $\mathbb{R}$ then satisfies intermediate value property.
(c) If f satisfies intermediate value proerty and $f^{-1}(\{r\})$ is closed $\forall\ r \in \mathbb{Q}$ then f is continuous on $\mathbb{R}$.
(d) None of the above.

(8) $f : [0, 1] \longrightarrow [0, 1]$ is defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \cap [0, 1] \\ 1 - x & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0, 1] \end{cases}$$

(a) f is continuous on $[0, 1]$ and does not satisfy intermediate value property.
(b) f satisfies intermediate value property but f is not continuous.
(c) f is continuous only at $x = \frac{1}{2}$ and $f[0, 1] = [0, 1]$. (d) None of the above.

(9) Cantor's Theorem is applicable in the following and $\cap_{n\in\mathbb{N}} F_n$ is a singleton set

(i) $X = [-1, 1], d$ usual distance, $F_n = [-\frac{1}{n}, \frac{1}{n}]$
(ii) $X = (0, 1), d$ usual distance, $F_n = [0, \frac{1}{n}]$
(iii) $X = \mathbb{R}, d$ discrete metric, $F_n = (0, \frac{1}{n})$
(iv) $X = [0, 1], d$ usual distance, $F_n = [1 - \frac{1}{n}, 1]$

(a) (i) and (ii) (b) (i) and (iv) (c) (i), (ii) and (iv) (d) None of these.

(10) Let d_1 and d_2 be equivalent metrics on X. Then
(a) (X, d_1) is bounded $\Longrightarrow (X, d_2)$ is bounded.
(b) (X, d_1) is complete $\Longrightarrow (X, d_2)$ is complete.
(c) (x_n) is a Cauchy sequence in $(X, d_1) \Longrightarrow (x_n)$ is a Cauchy sequence in (X, d_2).
(d) None of the above.

(11) Consider the following subspaces of $\mathbb{R}$ where distance in $\mathbb{R}$ is usual.
(i) $\mathbb{Q}$ *(ii)* $\mathbb{Z}$ *(iii)* $\{0\} \cup \{\frac{1}{n} : n \in \mathbb{N}\}$ *(iv)* $[-1, 1) \cup \mathbb{N}$. Then
(a) (i) and (iv) are complete .
(b) only (ii) is complete.
(c) (ii), (iii) and (iv) are complete.
(d) None of the above.

(12) Suppose $\| \ \|_1$ and $\| \ \|_2$ are equivalent norms on a normed linear space X. Then the statement which is not true is
(a) $(X, \| \ \|_1)$ is complete if and only if $(X, \| \ \|_2)$ is complete.
(b) (x_n) is a Cauchy sequence in $(X, \| \ \|_1$ if and only if (x_n) is a Cauchy sequence in $(X, \| \ \|_2)$.
(c) A is a bounded set in $(X, \| \ \|_1)$ if and only if A is bounded in $(X, \| \ \|_2)$.
(d) (a), (b) and (c) are not true.

(13) Consider the following subspaces of $(\mathbb{R}, d)$ where d is usual distance :
(i) $[0, \infty)$ (ii) $[0, 1] \cup [2, 3]$ (iii)$\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\}$ (iv) $\mathbb{Z}$ Then
(a) All the sub spaces are complete . (b) Only (i) is complete.
(c) Only (ii) is complete (d) Only (iii) is not complete.

(14) Let (X, d) be a complete metric space. A, B be complete subspaces of X such that $A \cap B \neq \emptyset$ then
(a) $A \cup B$ is a complete subspace of X but $A \cap B$ is not.
(b) $A \cap B$ is a complete subspace of X but $A \cup B$ is not.
(c) $A \cup B$ and $A \cap B$ are complete subspaces of X.
(d) None of the above.

(15) Consider the following subspaces under usual distance in $\mathbb{R}$.
(i) $\{\sqrt{2}, \sqrt{3}, \sqrt{5}\}$ (ii) $\{\sqrt{p} : p$ is a prime number$\}$ (iii) $\{x \in \mathbb{R} \setminus \mathbb{Q} : x \leq \sqrt{89}\}$ Then
(a) (i), (ii), (iii) are not complete.
(b) (i), (ii), (iii) are all complete.

(c) (i) and (ii) are complete and (iii) is not.
(d) None of the above.

(16) Consider the following subspaces of $(\mathbb{R}, d)$, where d is usual distance in $\mathbb{R}$. If $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$ are subspaces of $(\mathbb{R}, d)$. Then
(a) $\mathbb{N}, \mathbb{Z}$ and $\mathbb{Q}$ are complete , $\mathbb{R} \setminus \mathbb{Q}$ is not complete.
(b) $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are all complete.
(c) $\mathbb{N}, \mathbb{Z}$ are complete and $\mathbb{Q}, \mathbb{R} \setminus \mathbb{Q}$ are not complete.
(d) None of the above.

(17) Consider the space $C[a, b]$ with norms $\| \ \|_1$ and $\| \ \|_\infty$ where $\|f\|_1 = \displaystyle\int_a^b |f(x)| \, dx$ and $\|f\|_\infty = \sup\{|f(x)| x \in [a, b]\}$. Then
(a) $(C[a, b], \| \ \|_1)$ and $(C[a, b], \| \ \|_\infty)$ are complete.
(b) $(C[a, b], \| \ \|_1)$ is complete but $(C[a, b], \| \ \|_\infty)$ is not complete.
(c) $(C[a, b], \| \ \|_\infty)$ is complete but $(C[a, b], \| \ \|_1)$ is not complete.
(d) None of the above.

Topology of Metric Spaces: Practical 3.5
Complete Metric Spaces
DESCRIPTIVE QUESTIONS 3.5

(1) Check whether Cantor's Intersection theorem is applicable for the following examples. Also, find $\cap_{n \in \mathbb{N}} F_n$ in each case, where (F_n) is a sequence of subsets of $\mathbb{R}$ and the distance in $\mathbb{R}$ is usual.
(a) $Fn = (0, \infty)$ (b) $F_n = (0, \frac{1}{n})$ (c) $F_n = [1 - \frac{1}{n}, 2 + \frac{1}{n}]$

(2) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function which satisfies intermediate value property: for $a, b \in \mathbb{R}$ with $f(a) < \lambda < f(b)$, there exists c between a and b such that $f(c) = \lambda$. Further if $\{x \in \mathbb{R} : f(x) = r\}$ is closed set for each $r \in \mathbb{Q}$, then show that f is continuous on $\mathbb{R}$.

(3) Prove that there is no continuous function $f : [0, 1] \longrightarrow \mathbb{R}$ satisfying $x \in \mathbb{Q} \Longleftrightarrow f(x) \notin \mathbb{Q}$.

(4) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function such that $f^{-1}(\{x\})$ has exactly two points for each $x \in \mathbb{R}$. Show that f cannot be continuous on $\mathbb{R}$.

(5) Let h be defined on $[0, 1]$ (usual distance) as follows:

$$h(x) = \begin{cases} 0 & \text{if} \quad x \text{ is irrational.} \\ \frac{1}{n} & \text{if} \quad x \text{ is rational number} \frac{m}{n}, \text{with} (m, n) = 1 \\ 1 & \text{if} \quad x = 0 \end{cases}$$

Prove that h is continuous only at irrational points in $[0, 1]$.

(6) $f : [0,1] \longrightarrow [0,1]$ is defined by

$$f(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \cap [0,1] \\ 1-x & \text{if} \quad x \in \mathbb{R} \setminus \mathbb{Q} \cap [0,1] \end{cases}$$

Show that $f([0,1]) = [0,1]$ whereas f does not satisfy intermediate value property.

(7) Show that the equation $\cos x = x$ has at least one solution.

(8) Show that the equation $x^3 - 15x + 1 = 0$ has 3 solutions in the interval $[-4, 4]$.

(9) Show that the function $f(x) = (x-a)^2(x-b)^2 + x$ takes the value $(a+b)/2$ for some value of x.

(10) Let $f(x) = \tan x$; then $f(\pi/4) = 1$ and $f(3\pi/4) = -1$. But there is no $c \in (\pi/4, 3\pi/4)$ such that $f(c) = 0$. Explain why this does not contradicts Intermediate value property.

(11) Prove that if f, g are continuous on $[a,b]$ and $f(a) > g(a)$ and $f(b) < g(b)$ then there is a point $c \in (a,b)$ such that $f(c) = g(c)$.

(12) Use the intermediate value property to show that there is a square whose diagonal has length between r and $2r$ and has area equal to half the area of the circle of radius r.

(13) Show that a Cauchy sequence in a metric space (X,d) where, X is a finite set and d is any distance, is eventually constant. Hence show that (X,d) is complete.

(14) Show that Cauchy sequence in $(\mathbb{N}, d)$ (or $(\mathbb{Z}, d)$) where d is usual distance is eventually constant. Hence show that $(\mathbb{N}, d)$ (or $(\mathbb{Z}, d)$) is complete.

(15) Show that a Cauchy sequence in a discrete metric space (X,d) is eventually constant. Deduce that (X,d) is complete.

(16) Show that $(\mathbb{R}^2, d)$ is a complete metric space where $d(x,y) = 2|x_1 - y_1| + 3|x_2 - y_2|$ for $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

(17) Show that $(\mathbb{N}, d)$ is a complete metric space where for $m, n \in \mathbb{N}$,

$$d(m,n) = \begin{cases} 0 & \text{if } m = n \\ 1 + \dfrac{1}{m+n} & \text{if } m \neq n \end{cases}$$

(18) Let (X_1, d_1) and (X_2, d_2) be metric spaces and d be a metric on $X_1 \times X_2$ defined by $d\Big((x_1, x_2), (y_1, y_2)\Big) = \sqrt{d_1^2(x_1, y_1) + d_2^2(x_2, y_2)}$. Show that $\Big(x_n\Big) = \Big(x_1(n), x_2(n)\Big)$ in $X_1 \times X_2$ converges to (p_1, p_2) if and only if $x_1(n) \longrightarrow p_1$ and $x_2(n) \longrightarrow p_2$. Hence prove that if X_1, X_2 are complete, then $X_1 \times X_2$ is complete.

(19) Let (X_1, d_1) and (X_2, d_2) be complete metric spaces. Show that $\Big(X_1 \times X_2, d'\Big)$ and $\Big(X_1 \times X_2, d''\Big)$ are complete metric spaces where
$d'\Big((x_1, x_2), (y_1, y_2)\Big) = \alpha d_1(x_1, y_1) + \beta d_2(x_2, y_2)$
$d''\Big((x_1, x_2), (y_1, y_2)\Big) = \sqrt{\alpha d_1^2(x_1, y_1) + \alpha d_2^2(x_2, y_2)}$. where $\alpha, \beta > 0$.

(20) Show that the metric space $(C[0, 1], d_1)$ is not complete where $d_1(f, g) = \int_0^1 |f(x) - g(x)|\, dx$.
Hint: Consider the sequence $\{f_n\}$ in $C[0, 1]$ defined by

$$f_n(t) = \begin{cases} 0 & \text{if } 0 \le t \le \frac{1}{2} - \frac{1}{n} \\ nt - \dfrac{n}{2} + 1 & \text{if } \frac{1}{2} - \frac{1}{n} < t \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < t \le 1 \end{cases}$$

(21) Prove that $(0, 1)$ as a subspace of $(\mathbb{R}, d)$ (d being usual distance) is not complete but is complete as a subspace of $(\mathbb{R}, d_1)$ where d_1 is discrete metric.

(22) Show that $C[0, 1]$ with $\| \|_\infty$ defined as $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$ is complete.

Topology of Metric Spaces: Practical 3.6
Compact Metric Spaces
Objective Questions 3.6

. (Revised Syllabus 2018-19)

(1) Let (X, d) be a metric space and $K \subseteq X$. Then
(a) K is compact. (b) K is compact if K is closed.
(c) K is compact if K is bounded. (d) K is compact if K is finite.

(2) Let (X, d) be a metric space and (x_n) be a sequence in X such that $x_n \to x_0$ as $n \to \infty$. Then
(a) $\{x_n : n \in \mathbb{N}\}$ is a compact subset of X
(b) $\{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of X
(c) $\{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of X only if (x_n) is a sequence of distinct points.
(d) None of the above.

(3) Let $\{A_n\}$ be a family of compact subset of a metric space (X, d) such that $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$. Then
(a) $A_1 \cup \ldots \cup A_k, k \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} A_n$ are compact subsets of X.
(b) $A_1 \cap \ldots \cup A_k, k \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} A_n$ are compact subsets of X.
(c) $\cup_{n\in\mathbb{N}} A_n$ and $\cap_{n\in\mathbb{N}} A_n$ are compact subsets of X
(d) None of the above.

(4) Which of the following statements is false?
(a) A compact subset of a metric space is closed and bounded.
(b) A closed and bounded subset of a metric space is compact.
(c) A finite subset of a metric space is compact.
(d) A closed subset of a compact set in a metric space is compact.

(5) Which of the following are compact sunsets in the given metric space?
(a) $[0, 1]$ in $(\mathbb{R}, d_1)$ where d_1 is discrete metric.
(b) $\mathbb{N}$ in $(\mathbb{R}, d)$ where d is usual distance.
(c) $\left\{\left(\frac{1}{n}, \frac{(-1)^n}{n}\right) : n \in \mathbb{N}\right\} \cup \{(0, 0)\}$ in $(\mathbb{R}^2, d)$ where d is Euclidean distance.
(d) $[a, b] \cap \mathbb{Q}$ where a, b are irrational numbers in $(\mathbb{Q}, d)$ where d is usual distance.

(6) Consider the following subsets of $(\mathbb{R}^2, d)$, (d being Euclidean distance)

(i) $A = \{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$
(ii) $B = \{(x, y) \in \mathbb{R}^2 : y^2 = x\}$
(iii) $C = \{(x, y) \in \mathbb{R}^2 : 2x^2 + 3y^2 = 100\}$ Then

(a) A, B, C are compact.
(b) B, C are compact and A is not compact.
(c) Only A, B are compact.
(d) C is compact.

(7) Let (X, d) be a metric space and $x \in X$. Let $B[x, r]$ denote the closed ball $\{y \in Y : d(x, y) \leq r\}$ Then
(a) $B[x, r]$ is compact. (b) $B[x, r]$ is compact only if $r \leq 1$.
(c) $B[x, r]$ is compact if $X = \mathbb{R}$ and d is Euclidean distance. (d) None of the above.

(8) In the metric space $(\mathbb{Z}, d)$, ($\mathbb{Z}$ is the set of integers, d is usual distance), $K \subset \mathbb{Z}$
(a) if and only if K is closed. (b) if and only if K is bounded.
(c) if and only if K has a limit point. (d) if and only if $0 \in K$.

(9) Which of the following subsets of $\mathbb{R}^3$ are compact?
(a) $\{(x, y, x) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\}$ (b) $\{(x, y, x) \in \mathbb{R}^3 : x^2 - y^2 - z^2 = 1\}$
(c) $\{(x, y, x) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ (d) None of the above.

(10) Which of the following subsets of $\mathbb{R}^2$ is not compact? (distance being Euclidean) (a) The ellipse $\{(x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}, (a, b > 0)$
(b) The rectangular hyperbola $\{(x, y) \in \mathbb{R}^2 : xy = 1\}$
(c) The set $\{(x, y) \in \mathbb{R}^2 : x^2 + 2y^2 \leq 3^2\}$ (d) The set $\{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}$

(11) In the metric space $(\mathbb{R}, d)$ (d begin usual distance)
(a) $[0, 1] \cup [2, 3]$ is compact. (b) $[0, 1] \cup (2, 3)$ is compact.
(c) $[0, 1] \cup \{x \in \mathbb{N} : x \geq 3\}$ is compact. (d) $[0, 1] \cup [2, \infty)$ is compact.

(12) Consider the following subsets of $\mathbb{R}^2$ (distance being Euclidean).

(i) $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ (iii) $C = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \geq 1\}$
(ii) $B = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

(a) A, B, C are all compact. (b) A and B are compact, C is not compact.
(c) Only B is compact. (d) Only A is compact.

(13) Consider the following subsets of $(\mathbb{R}^n, d)$ (d being Euclidean distance)

$$
\begin{aligned}
A = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + x_2 + \ldots + x_n = 0\} \\
B = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} x_i^2 = 1\} \\
C = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^{n} |x_i| \leq n \text{ for } 1 \leq i \leq n\} \\
D = &\quad \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 = x_n = 0\}
\end{aligned}
$$

(a) A, B, C, D are compact sets. (b) Only B and C are compact sets.
(c) Only B, C and D are compact sets. (d) None of the above.

(14) Let A, B be compact subsets of $(\mathbb{R}, d)$, (d being usual). Then the following set is not compact.
(a) $A \times B$ in $(\mathbb{R}^2, d), d$ being Euclidean (b) $A \cup B$ in $\mathbb{R}$
(c) $A \cap B$ in $\mathbb{R}$ (provided $A \cap B \neq \emptyset$). (d) $A \setminus B$ in $\mathbb{R}$ (provided $A \setminus B \neq \emptyset$).

(15) Let (x_n) be a sequence in $[0, 1]$. Then, which of the following is **not true**.
(a) (x_n) has a convergent subsequence.
(b) (x_n) is bounded but may not be convergent.
(c) (x_n) is Cauchy.
(d) (x_n) may have subsequences converging to different limits.

(16) Let A be a compact subset of $\mathbb{R}$. Then
(a) $\overline{A}$ may not be compact. (b) A° may not be compact.
(c) ∂A may not be compact. (d) None of the above.

(17) Let A be a compact subset of $\mathbb{R}$. Then which of the following statements is not true
(a) A is complete. (b) A has a limit point in $\mathbb{R}$
(c) A is closed and bounded. (d) A° and ∂A are bounded.

Topology of Metric Spaces: Practical 3.6
Compact Metric Spaces
Descriptive Questions 3.6

(1) Using definition, show that $K = \left\{\dfrac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ is a compact subset of $(\mathbb{R}, d)$, where d is usual distance in $\mathbb{R}$. Also find a finite subcover of the open cover $\{B(\frac{1}{n}, \frac{1}{10})\}_{n\in\mathbb{N}}$ of K.

(2) Let (X, d) be a metric space and (x_n) be a sequence in X converging to x_0. Using definition, show that $K = \{x_n : n \in \mathbb{N}\} \cup \{x_0\}$ is a compact subset of (X, d)

(3) In the following examples, show that the set is not compact b considering the given open cover of the set:

(i) $C[a, b]$ in the metric space $(C[a, b], \| \|_\infty)$, $\|f\|_\infty =$ sup $\{|f(t)| : t \in [a, b]\}$. Show that the open cover $\{B(0, n)\}_{n\in\mathbb{N}}$ of $C[a, b]$ has no finite subcover. 0 being the constant zero function).

(ii) $(0, 1)$ in the metric space $(\mathbb{R}, d), d$ being the usual distance . Show that the open cover $\{(\frac{1}{n}, 1)\}_{n\in\mathbb{N}}$ of $(0, 1)$ has no finite subcover.

(iii) $\{\frac{1}{n} : n \in \mathbb{N}\}$ in the metric space $(\mathbb{R}, d), d$ being the usual distance. Show that the open cover $\{(\frac{1}{2n}, \frac{3}{2n})\}_{n\in\mathbb{N}}$ of $\{\frac{1}{n} : n \in \mathbb{N}\}$ has no finite subcover.

(iv) $[0,1]$ in the metric space $(\mathbb{R}, d_1), d_1$ being the discrete metric. Show that the open cover $\{B(x, \frac{1}{2})\}_{x\in[0,1]}$ has no finite subcover.

(4) Check if the following sets are compact in the given metric space. Justify your answer.

(i) $\{(x,y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$ in $(\mathbb{R}^2, d), d$ being Euclidean metric.

(ii) $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$ in $(\mathbb{R}^2, d), d$ being euclidean metric.

(iii) $\{n + \frac{1}{n} : n \in \mathbb{N}\}$ in $(\mathbb{R}, d), d$ being usual distance.

(5) Prove or disprove:

(i) A closed and bounded subset of a metric space is compact.

(ii) A closed ball $B[x, r]$ in a metric space is compact.

(iii) A compact set in a metric space is not open.

(iv) Interior and closure of a compact set are compact.

(6) Determine which of the following subsets of $(\mathbb{R}^2, d)$, where d is Euclidean distance is compact. Justify your answer.

(i) $\{(x,y) \in \mathbb{R}^2 : |x| + |y| \leq 1\}$

(ii) $\{(x,y) \in \mathbb{R}^2 : |x| \leq 1\}$

(iii) $\{(x,y) \in \mathbb{R}^2 : x \geq 1, 0 \leq y \leq \frac{1}{x}\}$

(iv) $\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}, (a, b > 0)$

(v) $\{(x,y) \in \mathbb{R}^2 : xy = 0\}$

(7) Let A, B be compact subsets of $\mathbb{R}$, distance being usual. Show that

(i) $A + B$ is a compact subset of $\mathbb{R}$.

(ii) $A \cup B$ is a compact subset of $\mathbb{R}$.

(iii) $A \times B$ is a compact subset of $(\mathbb{R}^2, d), d$ being Euclidean distance.

(8) Show that $A = (0,1]$ is not a compact subset of $(\mathbb{R}, d), d$ being Euclidean distance by

(i) exhibiting a sequence in A which has no convergent sequence.

(ii) exhibiting an infinite subset of A which has no limit point in A.

(9) Show that $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + 2x_2^2 + \cdots + nx_n^2 \leq (n+1)^2\}$ is a compact subset of $(\mathbb{R}^n, d), d$ being Euclidean.

(10) If A, B are disjoint non-empty subsets of (X, d) and A is closed, B is compact then show that $d(A, B) > 0$.

(11) Consider the set $A = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q}$ in a metric space $(\mathbb{Q}, d)$ where d is a usual metric from $\mathbb{R}$. Is the set A:

(i) closed and bounded in $(\mathbb{Q}, d)$?

(ii) compact in $(\mathbb{Q}, d)$?

(12) Show that the closed unit ball $B[0,1]$ in l^2 is not compact, where $l^2 := \{(x_n) \text{ in } \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^2 < \infty$ i.e. convergent $\}$; Further, for any $x = (x_n) \in l^2$; define $||x||_2 = \sqrt{\sum_{n=1}^{\infty} |x_n|^2}$. The metric on l^2 is the metric corresponding to this norm.

Topology of Metric Spaces: Practical 3.7
Miscellaneous.

. Revised Syllabus 2018-19

UNIT 1

(1) Define a metric space (X, d) and a normed linear space $(X, \| \ \|)$. Show that on a normed linear space $d : X \times X \longrightarrow \mathbb{R}$ defined by $d(x, y) = \|x - y\|$ is a metric.

(2) Define an open ball $B(x, r)$ in a metric space (X, d). Show that an open ball is an open set.

(3) State and prove Hausdorff property in a metric space (X, d)

(4) Show that in a metric space (X, d)

(i) an arbitrary union of open sets is open.

(ii) a finite intersection of open sets is open.

(5) Give an example to show that an arbitrary intersection of open sets need not be open.

(6) Let (X, d) be a metric space. Show that a subset G of X is open if and only if it is a union of open balls.

(7) Prove that any nonempty open subset of $\mathbb{R}$ (distance being usual) can be written as a finite or countable union of open mutually disjoint intervals.

(8) Let (X, d) be a metric space and $A \subseteq X$. Show that

(i) A° is an open set and is the larges open set contained in A.

(ii) A is open if and only if $A = A^\circ$

(9) Let (X, d) be a metric space and $A \subseteq X$. Show that

(i) $A \subseteq B \Longrightarrow A^\circ \subseteq B^\circ$

(ii) $A \cap B)^\circ = A^\circ \cap B^\circ$

(iii) $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$ and the inequality may be strict.

(10) Show that two metrics d and d' on a non-empty set X are equivalent if and only if for each $x \in X$, any open ball $B_d(x, r)$ contains an open ball $B_d(x, r,')$ for some $r' > 0$ and any open ball $B'_d(x, s)$ contains an open ball $B_d(x, s')$ for some $s' > 0$.

(11) Let (X, d) be a metric space and F be a subset of X. Show that the following statements are equivalent:

(i) $X \setminus F$ is open.

(ii) F contains all its limit points.

(12) Show that in a metric space (X, d) , the following statements are equivalent for a subset G of X.

(i) G is open

(ii) G does not contain any limit point of $X \setminus G$.

(13) Let (X, d) be a metric space and $A \subseteq X$. Show that

(i) $\overline{A}$ is a closed set.

(ii) A is closed if and only if $A = \overline{A}$.

(14) Let (X, d) be a metric space and $A, B \subseteq X$. Show that

(i) $A \subseteq B \Longrightarrow \overline{A} \subset \overline{B}$

(ii) $\overline{A \cup B} = \overline{A} \cup \overline{B}$

(iii) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$ and the inequality may be strict.

(15) Let (X, d) be a metric space and $A \subseteq X$. Show that $D(D(A)) \subseteq D(A)$ where $D(S)$ denotes the set of limit points of $S \subseteq X$. Hence show that $D(A)$ is closed.

(16) Bolzano-Weierstrass Theorem: Consider a metric space $(\mathbb{R}, d)$, where d is the usual metric. Prove that every infinite bounded subset of $\mathbb{R}$ must have a limit point in $\mathbb{R}$.

UNIT II

(1) Let (X, d) be a metric space and $A \subseteq X$. Show that $p \in \overline{A}$ if and only if there is a sequence of points in A converging to p.

(2) Let (X, d) be a metric space and A be a subset of X. Show that p is a limit point of A if and only if there is a sequence of distinct points converging to p.

(3) Prove: Every bounded sequence in $\mathbb{R}$ with usual metric, has a convergent subsequence.

(4) Show that a sequence (x_k) in $(\mathbb{R}^n, d)$ (where d is Euclidean distance) converge to a point $p = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$ if and only if $x_k^i \longrightarrow p_i$, for $1 \leq n_i$ in $\mathbb{R}$ with respect to the usual distance, where $x_k = (x_k^1, x_k^2, \cdots, x_k^n)$. Hence deduce that $(\mathbb{R}^n, d)$ is a separable metric space.

(5) Let (X, d) be a metric space and Y be a non-empty subset of X. Show that

(i) A subset G of Y is open in the subspace (Y, d) if and only of $G = V \cap Y$ where V is an open set in (X, d)

(ii) A subset F of Y is closed in the subspace (Y, d) if and only if $F = H \cap Y$ where H is closed set in (X, d).

(6) Let (X, d) be a metric space. Show that a convergent sequence in (X, d) is Cauchy. Give an example to show that the converse is not true. further show that a Cauchy sequence (x_n) in (X, d) is convergent if and only if it has a convergent subsequence.

(7) Show that the metric spaces (X, d_1) and (X, d_2) are equivalent if and only if (x_n) converges to p in (X, d_1) if and only if (x_n) converges to p in (X, d_2)

(8) Let (X, d) be a metric space . Show that a subset A of X is dense in X if and only if $G \cap A \neq \emptyset$ for each non-empty open subset G of X.

(9) Let (X, d) be a metric space. If $A \subseteq X$ is dense in X and B is a non-empty open subset of X then $\overline{A \cap B} = \overline{B}$.

(10) Prove that the metric space $(\mathbb{R}, d)$ is complete where d is the usual distance.

(11) Prove that the metric space $(\mathbb{R}^2, d)$ is complete where d is the Euclidean distance.

(12) Prove that the metric space $(\mathbb{C}, d)$ is complete with respect to the distance given by $d(z_1, z_2) = |z_1 - z_2|$

(13) Show that the metric space $(C[a, b], d)$ is complete where $d(f, g) = \sup\{|f(x) - g(x)| : x \in [a, b]\}$.

(14) Let (X, d) be a metric space and (Y, d_Y) be a subspace of (X, d). If (Y, d_Y) is complete then show that Y is closed.

(15) Let (X, d) be a complete metric space. If Y is a closed subspace of X then show that the subspace (Y, d_Y) is complete.

(16) State and prove Cantor's intersection theorem in a metric space (X, d).

(17) If in a metric space (X, d), for every decreasing sequence $\{F_n\}$ of non-empty closed sets with $d(F_n) \longrightarrow 0, \cap_{n\in\mathbb{N}} F_n$ is a singleton set then prove that (X, d) is complete.

(18) Nested Interval Theorem (As a particular case of Cantor's intersection theorem): Let $J_n = [a_n, b_n]$ be a sequence of intervals in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n \forall n \in \mathbb{N}$. Then show that $\bigcap\limits_{n\in\mathbb{N}} J_n \neq \emptyset$. If further we assume that $\lim\limits_{n\longrightarrow\infty} \ell(J_n) = 0$ then show that $\bigcap\limits_{n\in\mathbb{N}} J_n$ contains precisely one point.

As a consequence of Nested Interval Theorem:

(19) Show that set $\mathbb{R}$ of real numbers is uncountable.

(20) Density of rationals: Let $x < y$ be real numbers. Show that there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.

(21) Intermediate Value Theorem: Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous. Assume that $f(a)$ and $f(b)$ are of different signs, say, $f(a) < 0$ and $f(b) > 0$. Show that there exists $c \in (a, b)$ such that $f(c) = 0$.

UNIT III

(1) Show that a compact subset of a metric space is closed and bounded. Give an example to show that a closed and bounded subset of a metric space is not compact.

(2) Prove: A closed subset of a compact metric space is compact.

(3) Let (X, d) be a metric space and K is a compact subset of X. If F is a closed subset of X then show that $F \cap K$ is compact.

(4) Suppose (X, d) is a metric space and $\mathcal{C}$ is a non-empty collection of compact subsets of X then

(i) $\bigcap\limits_{K \in \mathcal{C}} K$ is a compact subset of X.

(ii) If $\mathcal{C}$ is finite then $\bigcup\limits_{K \in \mathcal{C}} K$ is a compact subset of X.

(5) Prove that a set A in a discrete metric space (X, d) is compact if and only if A is a finite set.

(6) Consider a metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that A is closed and bounded if and only if A satisfy Hein-Borel property. (A set is said to satisfy Hein-Borel property if every open conver of that set admits finite subcover).
Remark: The above result can be generalised to $(\mathbb{R}^n, d)$ as follows(without proof):
A subset A of $(\mathbb{R}^n, d)$ is closed and bounded if and only if it satisfy Hein-Borel property. Hence, $A \subset \mathbb{R}^n$ is compact if and only if it is closed and bounded.

(7) Consider a metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that A is closed and bounded if and only if A is sequentially compact. (A set A is said to be sequentially compact if every sequence in A has a covergent subsequence).

(8) Consider a metric space $(\mathbb{R}, d)$ where d is usual metric, $\emptyset \neq A \subset \mathbb{R}$. Prove that A is sequentially compact if and only if A satisfy Bolzano-Weierstrass property. (A set A is said to satisfy Bolzano-Weierstrass property if every non-empty, infinite subset of A has a limit point in A).