Paper – I

Unit III

Stochastic Process

A **stochastic** or **random process** can be defined as a collection of **random** variables that is indexed by some mathematical set, meaning that each **random** variable of the **stochastic process** is uniquely associated with an element in the set. Families of random variables which are functions of say, time, are known as Stochastic Process.

Examples:

1. Throwing a true dice. Let X_n is the outcome of nth throw. Then $\{X_n, n \geq 1\}$ is a family of random variables, such that for a distinct value of n (=1, 2,……), one gets a distinct random variableX_n.
$\{X_n, n \geq 1\}$ constitutes a Stochastic process, known as Bernoulli Process.

2. Consider that there are r cells and an infinitely large number of identical balls and the balls are drawn at random one by one and thrown in the cells, the ball being equally likely to go to any cell. Suppose X_n is the number of occupied cells after n throws. Then $\{X_n, n \geq 1\}$ constitutes a stochastic process.

3. Let X(t) is the number of telephone calls in an interval (0, t). $\{X(t),\}$ constitutes a Stochastic process.

4. X(t) might be the number of customers in a queue at time t. As time passes, customers will arrive and leave, and so the value of X(t) will change.

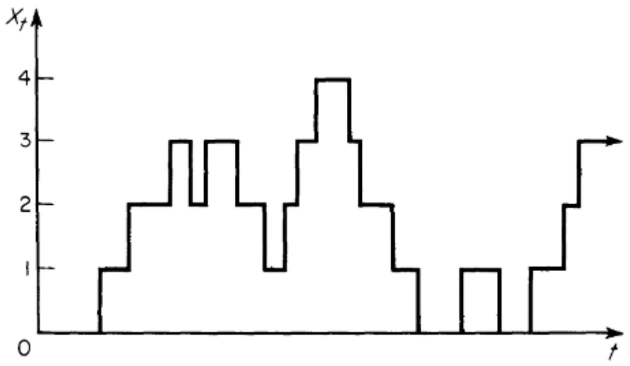


Figure 1.1 The number of customers, X_t, in a queue at time t

The set of all possible values of an individual random variable X_n *or* $X(t)$ *of a stochastic process is known as* its state space. The state space is discrete if it contains a finite or countably infinite number of points, otherwise it is continuous.

Changes in the value of X(t) are called transitions between its states.

Here we will consider some stochastic processes in continuous time and discrete state space. One such process is called **Poisson Process.**

For example, Incoming telephone calls, arrival of customers in a counter, occurrence of accidents etc.

Poisson processes can be seen in all walks of life. Here are some examples:

- At a drive-through pharmacy, the number of cars driving up to the drop off window in some interval of time on weekdays
- The number of hot dogs sold by Papaya King from 12pm to 4pm on Sundays.
- Failures of ultrasound machines in a hospital.
- The number of vehicles passing through some intersection from 8am to 11am on weekdays.
- Number of electrical pulses generated by a photo- detector that is exposed to a beam of photons, in 1 minute.

All occurrences are assumed to be of the same kind, and we are concerned with the total number *X (t)* of occurrences in an arbitrary time interval of length *t.*

Let us start from initial epoch (or instant) t=0, X(t) will represent the number of occurrences up to the epoch t.

Let $P_n(t)$ be the probability that the random variable assumes the value n, i.e.

$P_n(t) = P\{X(t) = n\}$. Eq. 1

The probability is a function of the time t. Since the only possible values of n are n= 0, 1, 2, 3,…….

$\sum_{n=0}^{\infty} P_n(t)$ =1

Thus $\{P_n(t)\}$ represents the probability distribution of the random variable Z(t) for every value of t. The family of random variables {X (t), t>0} is a stochastic process. Here the time t is continuous and X(t) is discrete.

The Poisson process has the following properties:

Let us choose an origin of time measurement and say that at epoch t > 0. The system is in state En if exactly n jumps occurred between 0 and t. Then $P_n(t)$ equals the probability of the state En at epoch t, but $P_n(t)$ may be described also as the transition probability from an arbitrary state Ej at an arbitrary epoch s to the state $E_{(j+n)}$ at epoch s + t.

Let us partition a time interval of unit length into N subintervals of length $h = \frac{1}{N}$. The probability of a jump within anyone among these subintervals equals

1- P (no jump) =1 - $P_o(h)$

So the expected number of subintervals containing a jump

= E (number of subintervals containing a jump)

= N [l-Po(h)] = h^{-1}[l-Po(h)]

One feels intuitively that as h → 0 this number will converge to the expected number of jumps within any time interval of unit length, and it is therefore natural to assume that there exists a number λ > 0 such that

$$h^{-1}[\text{l-Po(h)}] \to \lambda$$

$$\frac{1-P_0(h)}{h} \to \lambda$$

$$1-P_0(h) \to \lambda h$$

$$\Rightarrow\ -P_0(h) \to -1+\lambda h$$

$$\Rightarrow P_0(h) \to 1-\lambda h$$

$$\Rightarrow P_0(h) = 1-\lambda h + 0(h) \qquad Eq.2$$

o(h) stands for a quantity such that $\frac{0(h)}{h} \to 0\ as\ h \to 0$.

Expected number of subintervals containing more than one jump

= E (number of subintervals containing more than one jump)

= N [l-Po(h) - $P_1(h)$] Eq 3

The physical picture of the process that a jump always leads from a state Ej to the neighbouring state E_{j+1} and this implies that the expected number of subintervals (of length h) containing more than one jump should tend to 0.

Accordingly, we shall assume that as h → 0, N [l-Po(h) - $P_1(h)$]→ 0

$$\Rightarrow \frac{1-P_0(h)-P_1(h)}{h} \to 0$$

⇒ [l-Po(h) - $P_1(h)$]→ 0

Substituting $P_0(h) = 1-\lambda h + 0(h)$ from Eq. 2 in the above equation we get

$P_1(h) = \lambda h + 0(h)$

We now formulate the **Postulates for the Poisson process.**

1. **It is made up of a sequence of random variables** such that each variable represents the number of occurrences of some event, such as patients walking into a clinic during some interval of time.

2. **It is a stochastic process**. Each time you run the Poisson process, it will produce a different sequence of random outcomes as per some probability distribution which we will soon see.
3. **It is a discrete process**. The Poisson process's outcomes are *the number of occurrences* of some event in the specified period of time, which is undoubtedly an integer —i.e. a discrete number.
4. **Independence:** X(t) is independent of the number of occurrences (of the event E) in an interval prior to the interval (o, t); i.e. future changes in X(t) are independent of past changes.
5. **Homogeneity of time:** What this is means is that the number of events that the process predicts will occur in any given interval (0, t), is independent of the number in any other disjoint interval. For e.g. the number of people walking into the clinic from time zero (start of the observation) up through 10am, is independent of the number walking in from 3:33pm to 8:26pm, or from 11:00pm to 11:05pm and so on.

6. **Regularity:** Whatever the state E j at epoch t, the probability of one jump within an ensuing short time interval between t and t+h equals
$P_1(h) = \lambda h + 0(h)$
while the probability of more than one jump is o(h).
The probability of no occurrence is $P_0(h) = 1 - \lambda h + 0(h)$

Theorem: Under the postulates of Poisson process, X(t) follows Poisson distribution with mean λt.
$p_n(t) = \frac{e^{-(\lambda t)}(\lambda t)^n}{n!}$
$Proof$: Consider $p_n(t+h)\ for\ n\ \geq 0$
To prove this assume first n > 0 and consider the event that at epoch t+h the system is in state En. The probability of this event equals $P_n(t+h)$, and

the event can occur in three mutually exclusive ways. First, at epoch t the system may be in state En and no jump occurs between t and t+h.

The probability of this contingency is P n(t)Po(h) = P n(t) [l-λh + o(h)]

= $p_n(t)$ [l-λh] + o(h)

The second possibility is that at epoch t the system is in state E_{n-1} and exactly one jump occurs between t and t+h.

The probability for this is p_{n-1}(t) $p_1(h) =\ p_{n-1}$(t) [λh+ o(h)]

= p_{n-1}(t) λh+ o(h)

Any other state at epoch t requires more than one jump between t and t+h, and the probability of such an event is o(h).

Accordingly we must have

$p_n(t+h)$ = $p_n(t)$ [l-λh] + o(h) + p_{n-1}(t) λh+ o(h) + 0(h)

$p_n(t+h)$ = $p_n(t)$ [l-λh] + p_{n-1}(t) λh+ o(h) Eq. 1

$\Rightarrow\ p_n(t+h) = p_n(t) - p_n(t)\lambda h + p_{n-1}(t)\,\lambda h + o(h)$

$$\Rightarrow \frac{p_n(t+h)}{h} = \frac{p_n(t)}{h} - \lambda p_n(t) + \lambda p_{n-1}(t) + \frac{0(h)}{h}$$

$$\Rightarrow \frac{p_n(t+h)-p_n(t)}{h} = -\lambda p_n(t) + \lambda p_{n-1}(t) + \frac{0(h)}{h}$$

As h → 0, the last term tends to zero; hence the limit of the Jeft side exists and we get

$$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t) \qquad \text{Eq. 2}$$

For n = 0 Eq. 2 do not arise, and therefore from Eq. 1, we get

$p_0(t+h)$ = $p_0(t)$ [l-λh] + o(h)

$$\Rightarrow p_0(t+h) - p_0(t) = -\lambda h\, p_0(t) + 0(h)$$

$$\Rightarrow \frac{p_0(t+h)-p_0(t)}{h} = -\lambda p_0(t) + \frac{0(h)}{h}$$

As h → 0, the last term tends to zero; hence the limit of the left side exists and we get

$p_0'(t) \ = \ -\lambda p_0(t)$ Eq. 3

Case 1: Suppose that the process starts from the scratch at time 0 (or at the origin of epoch measurement), so that X(0) = 0, i.e.

$p_0(0) = 1 \; and \; p_n(0) = 0$ Eq. 4

Equations 2 and 3 are called **differential- difference equations**. Equations 2, 3 and 4 completely specifies the distribution. Their solution gives the probability distribution $p_n(t) \; of \; X(t)$.

The solution of Eq. 3 is given by $p_0(t) = Ce^{-\lambda t}$ Eq. 5

[Note: f(x) = C e^{-ax} ⇒ $\log f(x) = \ \log(C) - ax \Rightarrow \frac{f'(x)}{f(x)} = 0 - a \Rightarrow f'(x) = -a \; f(x)$]

Since $p_0(0) = 1, p_0(0) = Ce^{-\lambda\times 0} \Rightarrow 1 = C \times 1 \Rightarrow C = 1$

So from Eq. 5 we get, $p_0(t) = e^{-\lambda t}$

Substituting the value $p_0(t) = e^{-\lambda t}$ in Eq. 2, we get

$p_1'(t) \ = \ -\lambda p_1(t) + \lambda p_0(t)$

$p_1'(t) \ = \ -\lambda p_1(t) + \lambda \, e^{-\lambda t}$

$Let \; p_1(t) = \lambda t \; e^{-\lambda t}, p_1'(t) = \ -\lambda \; p_1(\mathrm{t}) + \lambda \; e^{-\lambda t}$

So $p_1(t)$= $\lambda t \; e^{-\lambda t} \ = \ \frac{e^{-\lambda t} \, (\lambda t)^1}{1!}$

By induction method, we get,

$$p_n(t) = \frac{e^{-(\lambda t)}(\lambda t)^n}{n!}$$

Case 2: Suppose that the process starts from “a” members at time 0. so that X(0) = a, i.e.

$p_a(0) = 1 \; and \; p_n(0) = 0$ if $n \neq a$ Eq. 6

Then $p_n(t) = \frac{e^{-\lambda t} \, (\lambda t)^{n-a}}{(n-a)!} \quad n \geq a$

$= 0 \qquad n < a$

Properties:

1. E(X(t)) = λt *and* $V(X(t)) = \lambda t$
2. If {X1(t)} and {X2(t)} are two independent Poisson processes with rates λ1 and λ2 respectively then X1(t) + X2(t) is a Poisson process with rate λ1 + λ2. Not a Poisson process.
3. If {X1(t)} and {X2(t)} are two independent Poisson processes with rates λ1 and λ2 respectively then X1(t) - X2(t) is not a Poisson process.
4. Interarrival time follows Exponential Distribution

THE PURE BIRTH PROCESS

The simplest generalization of the Poisson process is obtained by permitting the probabilities of jumps to depend on the actual state of the system.

Transitions of state may go only from n to n + 1. We can view the state of the system as the size of a population that can increase by a "birth" (n → n + 1)..

Suppose that whenever there are n people in the system, then: (i) new individuals enter the system at an exponential rate λn. Here we can say λ is not a constant, but it is a function of n. This leads us to the following Postulates.

(i) Direct transitions from a state Ej are possible only to E_{j+1}

(ii) If at epoch t the system is in state En the probability of a jump within an ensuing short time interval between t and t+h equals λ_nh + o(h), while the probability of more than one jump within this interval is o(h).

The salient feature of this assumption is that the time which the system spends in any particular state plays no role; there are sudden changes of state but no aging as long as the system remains within a single state.

Again let $p_n(t)$ be the probability that at epoch t the system is in state En·

The functions Pn(t) satisfy a system of differential equations which can be derived by the argument of the preceding section, with the only change that Eq. 1 is replaced by

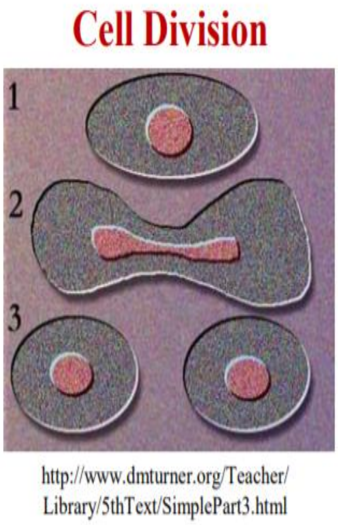
$p_n(t+h) = p_n(t)\,[1-\lambda_n h] + p_{n-1}(t)\,\lambda_{n-1} h + o(h)$

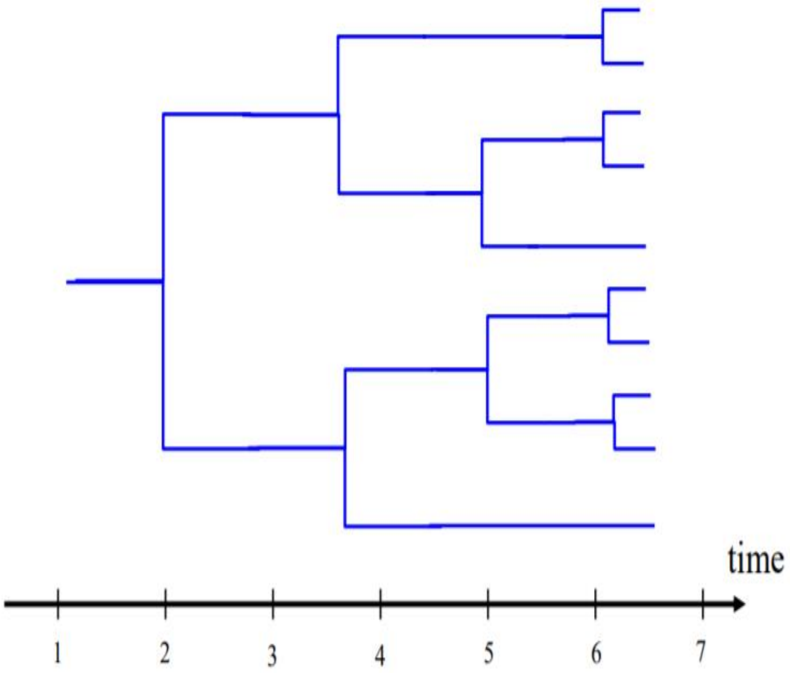
In this way we get the basic system of differential equations proceeding before and replacing λ by λ_n.

$$p_n'(t) \;=\; -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t) \qquad \text{Eq. 7}$$

$$p_0'(t) \;=\; -\lambda_0 p_0(t) \qquad \text{Eq. 8}$$

YULE FURRY PROCESS.





At any time there is some number of individuals in the population, and each individual gives birth to an offspring at constant rate λ, independently from the rest of the population. After a birth has happened, the parent and child evolve independently. In the notation of general birth processes, the birth rate when there are n individuals is $\lambda_n = n\lambda$. Consider a population of members which can (by splitting or otherwise) give birth to new members but cannot die. Assume that during any short time interval of length h each member has probability λh + o(h) to create a new one; the constant λ determines the rate of increase of the population. The probability that an increase takes place at some time between t and t+h equals nλh + o(h). The probability $p_n(t)$ that the population numbers exactly n elements therefore satisfies Eq 7 and 8 with $\lambda_n = n\lambda$, that is,

$$p_n'(t) \ = \ -n\lambda p_n(t) + (n-1)\lambda p_{n-1}(t) \qquad \text{Eq. 9}$$

$$p_0'(t) \ = \ -0\lambda(t) = 0$$

Denote the initial population size by n_0. It is easily verified that for $n \geq n_0 \geq 0$

$$\boxed{P_n(t) = \binom{n-1}{n-n_0} e^{-\lambda n_0 t}(1-e^{-\lambda t})^{n-n_0}} \qquad n = n_0, n_0+1, \ldots$$

Solution is negative binomial distribution.

Mean=$n_0 e^{\lambda t}$

Variance=$n_0 e^{\lambda t}(e^{\lambda t}-1)$

Problems:

1. Suppose the customers arrive at a bank according to Poisson process with rate of arrival 3 per minute. In an interval of 2 minutes

BIRTH AND DEATH PROCESS

The **birth–death process** (or **birth-and-death process**) is a special case of continuous-time Markov **process** where the state transitions are of only two types: "**births**", which increase the state variable by one and "**deaths**", which decrease the state by one. This suggests generalizing the model by permitting transitions from the state En not only to the next higher state E_{n+1} but also to the next lower state E_{n-1}.

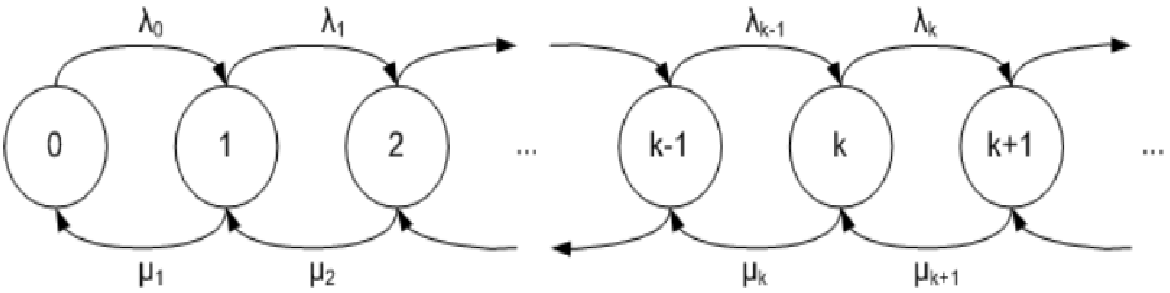
When a birth occurs, the process goes from state *n* to *n* + 1. When a death occurs, the process goes from state *n* to state *n* − 1. The process is specified by birth rates λ_i and death rates μ_i.

$$p_{ij} = \mu_i \qquad if\ j = i-1$$

$= \lambda_i \qquad\qquad if\ j = i + 1$

$= 1\text{-}\ \lambda_i - \mu_i \qquad if\ j = i$

= o otherwise



Postulates.

1. The system changes only through transitions from states to their nearest neighbours (from En to $E_{n+1}\ or\ E_{n-1}$ if n > 1, but from Eo to E_1 only).
2. If at epoch t the system is in state En the probability that between t and t+h the transition En ~ En+l occurs equals λ_nh + o(h), and the probability of En -~ En- 1 (if n > 1) equals μ_nh + o(h). The probability that during (t, t+h) more than one change occurs is o(h).

To calculate P n(t+h)= P(X(t+h) =n), note that the state En at epoch t+h is possible only under one of the following conditions: (1) At epoch t the system is in En and between t and t +h no change occurs; (2) at epoch t the system is in En- 1 and a transition to En occurs; (3) at epoch t the system is in En+1 and a transition to En occurs; (4) between t and t+h there occur two or more transitions. By assumption, the probabihty of the last event is o(h). The first three contingencies are mutually exclusive and their probabilities add. Therefore

$$P_n(t+h) = P_n(t)\{1-\lambda_n h-\mu_n h\}$$
$$+P_{n-1}(t)\{\lambda_{n-1}h\}+P_{n+1}(t)\{\mu_{n+1}h\}+o(h)$$

$$\Rightarrow p_n(t+h)$$
$$= p_n(t) - p_n(t)\lambda_n h - p_n(t)\mu_n h + p_{n-1}(t)\lambda_{n-1}h$$
$$+ p_{n+1}(t)\mu_{n+1}h + o(h)$$

$$\Rightarrow \frac{p_n(t+h)}{h} = \frac{p_n(t)}{h} - \lambda_n p_n(t) - p_n(t)\mu_n + \lambda_{n-1}p_{n-1}(t) + p_{n+1}(t)\mu_{n+1} + \frac{0(h)}{h}$$

$$\Rightarrow \frac{p_n(t+h)-p_n(t)}{h} = -\lambda_n p_n(t) - p_n(t)\mu_n + \lambda_{n-1}p_{n-1}(t) + p_{n+1}(t)\mu_{n+1} + \frac{0(h)}{h}$$

$$\Rightarrow p_n'(t) = -(\lambda_n+\mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + p_{n+1}(t)\mu_{n+1} \qquad \text{Eq. 10}$$

[As h → 0, the last term tends to zero]

For $n = 0$

$$P_0(t+h) = P_0(t)\{1-\lambda_0 h\} + P_1(t)\mu_1 h + o(h)$$

$$\Rightarrow \quad \boxed{P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)}$$

Eq. 11

If the initial state is Ei, the initial conditions are

$P_i(0) = 1 \; and \; p_n(0) = 0 \; for \; n \neq i.$ Eq. 12

The birth-and-death process is thus seen to depend on the infinite system of differential equations 10 and 11 together with the initial condition 12.

Note: 1. The coefficients {λn} and {μn} are called the birth and death rates respectively.

2. When μj = 0 for all j, the process is called a pure birth process.
3. We consider a Pure birth process (μi = 0) with birth rate= λn and initial state E0 .

$$p_n'(t) = -\lambda_n p_n(t) + \lambda_{n-1} p_{n-1}(t)$$
$$p_0'(t) = -\lambda_0 p_0(t)$$

4. We consider a Pure birth process (Poisson) (μi = 0) with constant λn = λ and initial state E0 .

$$p_n'(t) = -\lambda p_n(t) + \lambda p_{n-1}(t)$$
$$p_0'(t) = -\lambda p_0(t)$$

PURE DEATH PROCESS

We consider a Pure Death process with birth rate= λn=0 and initial state E0.

From Birth and Death process, we get,

$$P_n(t+h) = P_n(t)\{1 - \lambda_n h - \mu_n h\}$$
$$+ P_{n-1}(t)\{\lambda_{n-1} h\} + P_{n+1}(t)\{\mu_{n+1} h\} + o(h)$$

Eq. 13

Substituting λn = $\lambda_{n-1} = 0$

$$\Rightarrow p_n(t+h) = p_n(t) - p_n(t)\mu_n h + p_{n+1}(t)\mu_{n+1} h + o(h)$$

$$\Rightarrow \frac{p_n(t+h)}{h} = \frac{p_n(t)}{h} - p_n(t)\mu_n + p_{n+1}(t)\mu_{n+1} + \frac{0(h)}{h}$$

$$\Rightarrow \frac{p_n(t+h) - p_n(t)}{h} = -p_n(t)\mu_n + p_{n+1}(t)\mu_{n+1} + \frac{0(h)}{h}$$

$$\Rightarrow p_n'(t) = -p_n(t)\mu_n + p_{n+1}(t)\mu_{n+1}$$ Eq. 14

[As h → 0, the last term tends to zero]

From Birth and Death process, we get,

For $n = 0$

$$P_0(t+h) = P_0(t)\{1 - \lambda_0 h\} + P_1(t)\mu_1 h + o(h)$$

$$\Rightarrow \boxed{P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)}$$

Substituting λ0 = 0

$p_0'(t) \ = \ \mu_1 p_1(t)$ Eq. 15

Note:

1. **Poisson Death process with μ_n= μ**

In Eq 14 and 15 substituting $\mu_n \ = \mu$, we get

$p_n'(t) = \quad -p_n(t)\mu + p_{n+1}(t)\mu$ Eq. 16

$p_0'(t) \ = \ \mu p_1(t)$ Eq. 17

Equations 16 and 17 are called **differential- difference equations**.

Denote the initial population size by n_0. It is easily verified that for $n \geq n_0 \geq 0$

Solving the differential equations we get,

Pn (t) = $\frac{e^{-\mu t}\,(\mu t)^{n_0-n}}{(n_0-n)!}$ $n_0 \geq n$ and n= 1, 2, 3, ….n_0

P0(t) = 1 - $\sum_{n=1}^{n_0} p_n(t)$

2. **Death process with μ_n= nμ,**

In Eq 14 and 15 substituting $\mu_n \ = n\mu$, we get

$p_n'(t) = \quad -np_n(t)\mu + (n+1)p_{n+1}(t)\mu$

$p_0'(t) \ = \ \mu p_1(t)$

Suppose initially, there are n0 individuals, that is X(0) = n0.

By solving the equations we get,

$p_n(t) = \binom{n_0}{n} e^{-n\mu t}(1-e^{-\mu t})^{n_0-n};$ n=$n_0, n_0 - 1, \ldots \ldots .. 1, 0$.

the mean of the pure death process is

$$E(X(t)) = n_0 e^{-\mu t},$$

and that the variance is

$$V(X(t)) = n_0 e^{-\mu t}(1 - e^{-\mu t}).$$

LINEAR GROWTH MODEL

From Birth and Death process, we get

$$P_n(t+h) = P_n(t)\{1 - \lambda_n h - \mu_n h\}$$
$$+ P_{n-1}(t)\{\lambda_{n-1} h\} + P_{n+1}(t)\{\mu_{n+1} h\} + o(h)$$

$$\boxed{P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t)}$$

Let us consider here both birth and death process occurs linearly. Suppose the probability that a member gives birth to a new member in a small interval of time h is $\lambda h + 0(h)$ and the probability that a member dies is $\mu h + 0(h)$. Then if n members are present at a time t, then $\lambda_n = n\lambda \; and \; \mu_n = n\mu.$

Where n $\geq 1 \; and \; \lambda_0 = \mu_0 = 0.$

$\Rightarrow p_n'(t) = -(\lambda_n + \mu_n)p_n(t) + \lambda_{n-1}p_{n-1}(t) + p_{n+1}(t)\mu_{n+1}$ [Refer Eq. 10]

Substituting $\lambda_n = n\lambda \; and \; \mu_n = n\mu.$

We get the differential difference equation as

$$P_n'(t) = -(\lambda + \mu)nP_n(t) + \lambda(n-1)P_{n-1}(t) + \mu(n+1)P_{n+1}(t)$$

$p_0'(t) = \mu p_1(t)$

Define Mean by $M(t) = \sum_{n=1} nP_n(t)$

and consider $M'(t) = \sum_1^{\infty} nP_n'(t)$.

$$M'(t) = -(\lambda+\mu)\sum_1^{\infty} n^2 P_n(t) + \lambda\sum_1^{\infty} (n-1)n\, P_{(n-1)}(t)$$
$$+\ \mu\sum_1^{\infty} (n+1)nP_{n+1}(t)$$

Write $(n-1)n = (n-1)^2 + (n-1), \quad (n+1)n = (n+1)^2 - (n+1)$

$$M'(t) = -\ (\lambda+\mu)\sum_1^{\infty} n^2 P_n(t)$$
$$+\ \lambda\sum_1^{\infty}(n-1)^2 P_{n-1}(t) + \mu\left[\sum_1^{\infty}(n+1)^2 P_{n+1}(t) + 1\cdot P_1(t)\right]$$
$$+\ \lambda\sum_1^{\infty}(n-1)P_{n-1}(t) - \mu\left[\sum_1^{\infty}(n+1)P_{n+1}(t) + P_1(t)\right]$$

$$\Rightarrow M'(t) = \lambda\sum_1^{\infty} nP_n(t) - \mu\sum_1^{\infty} nP_n(t)$$
$$= (\lambda-\mu)M(t)$$

$\boxed{M(t) = n_0 e^{(\lambda-\mu)t}}$ if $P_{n_0}(0) = 1$

$$\boxed{M(t) = n_0 e^{(\lambda-\mu)t}}$$

$M(t) \to 0$ or ∞ depending on $\lambda < \mu$ or $\lambda > \mu$.

Similarly if $M_2(t) = \sum_1^\infty n^2 P_n(t)$ one can show

$$M_2'(t) = 2(\lambda - \mu)M_2(t) + (\lambda + \mu)M(t)$$

and when $\lambda > \mu$, the variance is

$$\boxed{n_0 e^{2(\lambda-\mu)t}\left\{1 - e^{(\mu-\lambda)t}\right\}\frac{\lambda+\mu}{\lambda-\mu}}$$