Topology of Metric Spaces and Real Analysis: Practical 3.1
Continuous Functions on Metric Spaces.
Objective Questions 3.1

. (Revised Syllabus 2018-19)

(1) Let d be the usual distance in $\mathbb{R}$. For any $A \subseteq \mathbb{R}, d_A : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $d_A(x) = \inf\{d(x,a) : a \in A\}$. Then
(a) $d_{\mathbb{R}}, d_{\mathbb{R}\setminus\mathbb{Q}}$ are not continuous on $\mathbb{R}$ and $d_{\mathbb{Q}}(x) > 0 \quad \forall x \in \mathbb{R} \setminus \mathbb{Q}$.
(b) $d_{\mathbb{Q}} \equiv 0$ and $d_{\mathbb{R}\setminus\mathbb{Q}} \equiv 0$ on $\mathbb{R}$ and $d_{\mathbb{Q}}, d_{\mathbb{R}\setminus\mathbb{Q}}$ are continuous on $\mathbb{R}$.
(c) $d_{\mathbb{R}}, d_{\mathbb{R}\setminus\mathbb{Q}}$ are continuous on $\mathbb{R}$ and $d_{\mathbb{R}\setminus\mathbb{Q}}(x) > 0 \quad \forall x \in \mathbb{Q}$.
(d) None of the above.

(2) Let d denote the usual distance in $\mathbb{R}$ and for $A \subseteq \mathbb{R}$, let

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}.$$

Then
(a) χ_A is continuous on $\mathbb{R}$ if and only if A is an open subset of $\mathbb{R}$.
(b) χ_A is continuous on $\mathbb{R}$ if and only if A is a closed subset of $\mathbb{R}$.
(c) χ_A is continuous on $\mathbb{R}$ if and only if $A = \emptyset$ or $A = \mathbb{R}$.
(d) None of the above.

(3) Consider the metrics d and d_1 on $\mathbb{N}$, where d is the induced distance from $\mathbb{R}$ with usual distance and $d_1(m,n) = \left|\frac{1}{m} - \frac{1}{n}\right|$ for $m, n \in \mathbb{N}$. Let $i : \mathbb{N} \longrightarrow \mathbb{N}$ denote the identity map on $\mathbb{N}$. Then
(a) $i : (\mathbb{N}, d) \longrightarrow (\mathbb{N}, d_1)$ is continuous but $i : (\mathbb{N}, d_1) \longrightarrow (\mathbb{N}, d)$ is not continuous.
(b) $i : (\mathbb{N}, d) \longrightarrow (\mathbb{N}, d_1)$ is not continuous.
(c) $i : (\mathbb{N}, d_1) \longrightarrow (\mathbb{N}, d)$ is not continuous.
(d) None of the above.

(4) Let d_1 and d_2 be equivalent metrics on X and (Y, d) be any metric space. If $f : (X, d_1) \longrightarrow (Y, d)$ and $g : (Y, d) \longrightarrow (X, d_1)$ are continuous maps on X and Y respectively, then
(a) $f : (X, d_2) \longrightarrow (Y, d)$ is continuous, but $g : (Y, d) \longrightarrow (X, d_2)$ may not be continuous
(b) $f : (X, d_2) \longrightarrow (Y, d)$ may not be continuous, but $g : (Y, d) \longrightarrow (X, d_2)$ is continuous
(c) $f : (X, d_2) \longrightarrow (Y, d)$ and $g : (Y, d) \longrightarrow (X, d_2)$ are continuous on X and Y respectively.
(d) None of the above.

(5) Let $A = \{x \in \mathbb{R} : \sin x = \frac{1}{2}\}$, the distance in $\mathbb{R}$ being usual. Then
(a) A is an infinite closed set. (b) A is a finite closed set.
(c) A is an open set. (d) None of the above.

(6) Let (X, d) and (Y, d') be metric spaces and $f, g : X \longrightarrow Y$ be continuous maps. If $A \subseteq X$ such that $f(x) = g(x) \quad \forall\, x \in A$, then the statement which is **not true** is

(a) $f(x) = g(x) \quad \forall x \in A^\circ$
(b) $f(x) = g(x) \quad \forall\ x \in \overline{A}$
(c) $f(x) = g(x) \quad \forall\ x \in \delta A$ where δA is the boundary of A
(d) All the above statements are false

(7) Let (X, d) and (Y, d') be metric spaces and $f, g : X \longrightarrow Y$ be continuous maps. Let $A = \{x \in X : f(x) = g(x)\}$. Then
(a) A is a dense subset of X. (b) A is a closed subset of X.
(c) A is an open subset of X. (d) None of the above.

(8) Let d denote the usual distance in $\mathbb{R}$ and d_1 denote the discrete metric on $\mathbb{R}$. Let $i : (\mathbb{R}, d_1) \longrightarrow (\mathbb{R}, d)$ be the identity map. Then
(a) $\overline{i(\mathbb{Q})} \subseteq i(\overline{\mathbb{Q}})$. (b) $i^{-1}(\overline{\mathbb{Q}}) \subseteq \overline{i^{-1}(\mathbb{Q})}$.
(c) $\overline{i^{-1}(\mathbb{Q})} \subseteq i^{-1}(\overline{\mathbb{Q}})$. (d) None of the above.

(9) Let (X, d) and (Y, d') be metric spaces and $f : X \longrightarrow Y$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a family of closed subsets of X. Then the statement which is **not true** is
(a) If f is continuous on A_1 and A_2, then f is continuous on $A_1 \cup A_2$.
(b) If f is continuous on each A_n, then f is continuous on $\bigcup\limits_{n=1}^{\infty} A_n$.
(c) If f is continuous on each A_n, then f is continuous on $A = \bigcap\limits_{n\in\mathbb{N}} A_n$, provided $A \neq \emptyset$
(d) None of the above.

(10) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ (the distance in $\mathbb{R}$ and $\mathbb{R}^2$ are Euclidean) be defined by $f(x, y) = |x|$. Then
(a) f is not continuous at $(x, 0)$ for each $x \in \mathbb{Z}$.
(b) f is not continuous at $(0, 0)$.
(c) f is continuous on $\mathbb{R}^2$
(d) None of the above.

(11) $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are any maps, such that $f \circ g$ and $g \circ f$ are continuous (distance being usual). Then
(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous
(b) $f \circ g = g \circ f$
(c) At least one of f and g is coninuous.
(d) Neither f nor g may be continuous.

(12) Let (X, d) be a metric space where X is a finite set and (Y, d') be any metric space. Let $f : X \longrightarrow Y$. Then the statement which is **not true** is
(a) f is continuous on X
(b) $f(X)$ is bounded.
(c) If A is open in X, $f(A)$ is open in Y
(d) If B is closed in $Y, f^{-1}(B)$ is closed in X.

(13) Let (X, d) be a compact metric space and $f : X \longrightarrow (0, \infty)$. (distance usual) be a continuous function. If $\inf\{f(x) : x \in X\} = m$, then
(a) m may be 0 (b) $m = 0$ (c) $m > 0$ (d) m may be negative

(14) Let (X, d) be a finite metric space, $|X| > 2$. If $f : X \longrightarrow \mathbb{R}$ (usual distance) is a continuous function, then
(a) $|f(X)| \geq 2$ (b) $f(X) = [m, M]$ for some $m, M \in \mathbb{R}$
(c) f is a constant function. (d) None of the above.

(15) Let (X, d) be a metric space. If $f, g \in C(X, \mathbb{R})$, then
(a) $f + g \in C(X, \mathbb{R})$, but $f - g$ may not be in $C(X, \mathbb{R})$.
(b) $f + g, f - g$ and $2f \in C(X, \mathbb{R})$.
(c) $f + g, f - g \in C(X, \mathbb{R})$, but $2f$ may not be in $C(X, \mathbb{R})$
(d) $f + g, f - g \in C(X, \mathbb{R})$, but fg may not be in $C(X, \mathbb{R})$

(16) Let $X_1 = [0, 1]; Y_1 = [0, \infty); X_2 = (0, 1) \cup (2, 3), Y_2 = (0, 1); X_3 = (0, 1), Y_3 = \{0, 1\}$. Then there exists a continuous onto function from $X_i \longrightarrow Y_i$ when
(a) $i = 1, 2, 3$ (b) $i = 1, 2$ (c) $i = 2$ (d) $i = 3$

(17) Consider the map $L : C[0, 1] \longrightarrow \mathbb{R}$ (usual distance) defined by $L(f) = \int_0^1 f(t)\, dt$. Then,
(a) $L : (C[0, 1], \|\ \|_1) \longrightarrow \mathbb{R}$ is continuous but $L : (C[0, 1], \|\ \|_\infty) \longrightarrow \mathbb{R}$ is not continuous.
(b) $L : (C[0, 1], \|\ \|_\infty) \longrightarrow \mathbb{R}$ is not continuous.
(c) $L : (C[0, 1], \|\ \|_1) \longrightarrow \mathbb{R}$ and $L : (C[0, 1], \|\ \|_\infty) \longrightarrow \mathbb{R}$ are both not continuous.
(d) None of the above.

(18) Consider the map $\phi : C[0, 1] \longrightarrow \mathbb{R}$ defined by $\phi(f) = f(0)$. Then
(a) $\phi : (C[0, 1], \|\ \|_\infty) \longrightarrow \mathbb{R}$ is not continuous.
(b) $\phi : (C[0, 1], \|\ \|_\infty) \longrightarrow \mathbb{R}$ is continuous.
(c) $\phi : (C[0, 1], \|\ \|_\infty) \longrightarrow \mathbb{R}$ and $\phi : (C[0, 1], \|\ \|_1) \longrightarrow \mathbb{R}$ are not continuous.
(d) None of the above.

(19) Let (X, d) be a metric space and $f \in C(X, \mathbb{R})$ be a bounded function. Then f

(a) attains both bounds.
(b) attains at least one bound.
(c) may not attain either bound.
(d) None of the above.

(20) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function (distance usual) and $A \subseteq \mathbb{R}$. Consider the following statements:
(i) If A is closed and bounded, $f(A)$ is closed and bounded.
(ii) If A is closed, $f(A)$ is closed. (iii) If A is bounded, $f(A)$ is bounded.
(a) (i), (ii), (iii) are true statements.
(b) (i) and (iii) are true, (ii) is not true.
(c) Only (i) is true.
(d) (i) and (ii) are true, (iii) is not true.

(21) Let (X, d) be a compact metric space and $f : X \longrightarrow \mathbb{R}$ is continuous. Let (x_n) be a sequence in X. Which statement is false?
(a) If (x_n) is convergent $(f(x_n))$ is convergent.
(b) If (x_n) is Cauchy, $(f(x_n))$ is Cauchy.
(c) $(f(x_n))$ has convergent subsequence.
(d) None of (a), (b), (c) are false.

Topology of Metric Spaces and Real Analysis: Practical 3.1
Continuous Functions on Metric Spaces
Descriptive Questions 3.1

(1) Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous function (with respect to usual distance). Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $h(x, y) = (f(x), g(y))$. Show that $h : (\mathbb{R}^2, d) \longrightarrow (\mathbb{R}^2, d)$ is Continuous where d is Euclidean distance.

(2) Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be continuous map. Show that $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $g(x, y) = f(x + y, x - y)$ is continuous.

(3) Show that $i : (\mathbb{R}, d) \longrightarrow (\mathbb{R}, d_1)$ where d is usual distance in $\mathbb{R}$ and d_1 is discrete metric on $\mathbb{R}$ is not continuous where i is the identity map on $\mathbb{R}$.

(4) Let (X, d) be a metric space and let $A \subseteq X$, If $d_A : X \longrightarrow \mathbb{R}$ is defined by $d_A(x) = d(x, A)$. Show that d_A is continuous.

(5) $X = M_2(\mathbb{R})$ and $\|A\| = \left(\sum\limits_{1 \leq i,j \leq 2} a^2_{(ij)} \right)^{\frac{1}{2}}$. Show that $f : X \longrightarrow \mathbb{R}$ (distance usual) defined by $f(A) = \det A$ is continuous. Hence show that
(i) $(GL)_2(\mathbb{R})$ is an open subset of X.
ii) $(SL)_2(\mathbb{R})$ is a closed subset of X.

(6) Prove or disprove:
a) If (X, d) and (Y, d') are metric spaces and $f : X \longrightarrow Y$ is a continuous bijective map, then for any open ball B in $(X, d), f(B)$ is an open ball (Y, d').
b) Let (X, d) and (Y, d') be metric spaces. If (X, d) is complete and $f : X \longrightarrow Y$ is continuous and onto, then (Y, d') is complete.

(7) Let (X, d) and (Y, d') be metric spaces. Prove that $f : X \longrightarrow Y$ is continuous on X if and only if f is continuous on each compact subset of X.

(8) Let A, B be two compact subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that $d(A, B) > 0$ and $\exists\ a \in A, b \in B$ such that $d(A, B) = d(a, b)$.

(9) Let $K \subseteq \mathbb{R}^n$ be such that any continuous function from$K$ to $\mathbb{R}$ be bounded. Show that K is compact.

(10) Show that $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is a compact subset of $\mathbb{R}^2$, distance being Euclidean.

(11) Let $f : X \longrightarrow (0, \infty)$ be a continuous function, where (X, d) is a compact metric space. Show that $\exists \ \epsilon > 0$ such that $f(x) \geq \epsilon, \forall x \in X$.

(12) $\psi : (C[0, 1], \| \ \|_\infty) \longrightarrow \mathbb{R}$ (usual distance) defined by $\psi(f) = f(0)$ is continuous.

(13) $L : (C[0, 1], \| \ \|_\infty) \longrightarrow \mathbb{R}$ (usual distance) defined by $L(f) = \displaystyle\int_0^1 f(t)\, dt$ is continuous.

Topology of Metric Spaces and Real Analysis: Practical 3.2
Uniform Continuity and Fixed Point Theorem
Objective Questions 3.2

. (Revised Syllabus 2018-19)

(1) $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ defined by, $f(x) = \dfrac{1}{x}$ for $x \neq 0$ is uniformly continuous on
(a) $(0, 1)$ (b) $(0, \infty)$ (c) $[1, \infty)$ (d) None of these.

(2) $f(x) = \dfrac{1}{1 + x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on
(a) $[0, 1]$ but not on $[0, \infty)$ (c) $\mathbb{R}$
(b) $[1, \infty)$ but not on $[0, \infty)$ (d) None of these.

(3) Let $A \subseteq \mathbb{R}$. If $f, g : A \longrightarrow \mathbb{R}$ are uniformly continuous on A, then
(a) $f + g$ is uniformly continuous on A but $f \cdot g$ may not be uniformly continuous on A.
(b) $f + g$ and $f \cdot g$ are uniformly continuous on A.
(c) Neither $f + g$ nor $f \cdot g$ may be uniformly continuous on A.
(d) None of the above.

(4) Consider the following functions (distance in $\mathbb{R}$ is usual):

(i) $f : [0, 2\pi] \longrightarrow \mathbb{R}, f(x) = x \sin x$
(ii) $f : (0, 1) \longrightarrow \mathbb{R}, f(x) = \frac{1}{x}$
(iii) $f : [0, 1] \times [0, 1] \longrightarrow \mathbb{R}, f(x, y) = x + y$ (distance in $\mathbb{R}^2$ Euclidean)

(a) (i), (ii), (iii) are uniformly continuous.
(b) (i) and (iii) are uniformly continuous, (ii) is not.
(c) Only (i) is uniformly continuous.
(d) Only (iii) is uniformly continuous.

(5) Suppose A and B are closed subsets of $\mathbb{R}$ and $f : A \cup B \longrightarrow \mathbb{R}$ is uniformly continuous on A as well as B. Then,
(a) f is uniformly continuous on $A \cup B$.
(b) f is uniformly continuous on $A \cup B$ if $A \cap B = \emptyset$.
(c) f may not be uniformly continuous on $A \cup B$.
(d) None of the above.

(6) $f : [0, \infty) \longrightarrow \mathbb{R}$ defined by $f(x) = \sqrt{x}$ is
(a) continuous on $[0, \infty)$ but not uniformly continuous on $[0, \infty)$.
(b) uniformly continuous on $[0, 1]$ but not on $[0, \infty)$.
(c) uniformly continuous on $[0, \infty)$.
(d) None of the above.

(7) If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are uniformly continuous on $\mathbb{R}$, then
(a) The product $f \cdot g$ uniformly continuous on $\mathbb{R}$.
(b) The composites $f \circ g$ and $g \circ f$ uniformly continuous on $\mathbb{R}$.
(c) f^2 and g^2 are uniformly continuous on $\mathbb{R}$.
(d) None of the above.

(8) Let (X, d) be a metric space and A be a non-empty subset of X. Then $d_A : X \longrightarrow \mathbb{R}$ defined by $d_A(x) = d(x, A) = \inf\{d(x, a) : a \in A\}$ is

(a) continuous on A but not on X.
(b) uniformly continuous on X.
(c) not uniformly continuous on X.
(d) None of these.

(9) Let (X, d) and (Y, d') be a metric spaces and $f : X \longrightarrow Y$. Suppose (x_n) is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y if

(a) f is continuous on X.
(b) f is uniformly continuous on X.
(c) X and Y are complete.
(d) None of these.

(10) Let $A \subseteq \mathbb{R}, A$ is bounded but not closed. Then
(a) Any continuous function from A to $\mathbb{R}$ is bounded.
(b) Any continuous function from A to $\mathbb{R}$ is uniformly continuous.
(c) Any continuous, bounded function from A to $\mathbb{R}$ attains bounds.
(d) None of the above.

(11) Let (X, d) and (Y, d') be metric spaces and $f : X \longrightarrow Y$ a uniformly continuous function. Then the statement which is not true is
(a) Given a bounded subset A of X, $f(A)$ need not be a bounded subset of Y.
(b) If $\{x_n\}$ is a Cauchy sequence in X, then $\{f(x_n)\}$ is a Cauchy sequence in Y.
(c) If $\{f(x_n)\}$ is a Cauchy sequence in Y, $\{x_n\}$ is a Cauchy sequence in X.
(d) If $\{x_n\}$ is convergent, then $\{f(x_n)\}$ is convergent.

(12) Consider the following maps:

(i) $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that f is differentiable and $|f'(x)| \leq M \ \forall \ x \in \mathbb{R}$.

(ii) A linear transformation $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$.

(iii) A map $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying Lipchitz condition namely $\exists \ M \geq 0$ such that $|f(x) - f(y)| \leq M|x - y| \quad \forall \ x, y \in \ \mathbb{R}$.

Then
(a) (i) and (iii) are uniformly continuous.
(b) (i), (ii) and (iii) are uniformly continuous.
(c) only (iii) is uniformly continuous.
(d) None of above.

(13) Which of the following real valued functions are uniformly continuous on the give sets.

(i) $f(x) = \dfrac{1}{x}$ on $(0, 1)$. (ii) $f(x) = x^{\frac{1}{3}}$ on $[0, 1]$.

(a) Only (i) (b) only (ii) (c) both (i) and (ii) (d) Neither (i) nor (ii).

Topology of Metric Spaces and Real Analysis: Practical 3.2
Uniform Continuity and Fixed Point Theorem
Descriptive Questions 3.2

(1) Show that the function $f(x) = \dfrac{1}{1+x^2}$ for $x \in \mathbb{R}$ is uniformly continuous on $\mathbb{R}$.

(2) Prove or disprove:
If $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are uniformly continuous on a nonempty set $A \subseteq \mathbb{R}$ then the product function $f \cdot g$ is uniformly continuous on A.

(3) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that $f'(x)$ exists $\forall\ x \in\ \mathbb{R}$ and $\exists$ a constant M such that $|f'(x)| \leq M \quad \forall\ x \in\ \mathbb{R}$, then show that f is uniformly continuous on $\mathbb{R}$.

(4) If $(X, d), (Y, d')$ are metric spaces, then prove that any Lipschitz function $f : (X, d) \longrightarrow (Y, d)$ is uniformly continuous. Hence, deduce that $\sin x, \cos x$ are uniformly continuous on $\mathbb{R}$.

(5) Let $A = (0, 1] \subset \mathbb{R}$. Define $d_A : \mathring{A} \to \mathbb{R}$ as $d_A(x) = d(x, A)$. Draw graph of d_A. Further, prove that if (X, d) is a metric space and $A \subseteq X$ then $d_A : X \longrightarrow \mathbb{R}$ defined by $d_A(x) = d(x, A)$ for $x \in\ X$ is uniformly continuous on X.

(6) Let $f : [a, b] \longrightarrow [a, b]$ be differentiable and $|f'(x)| \leq c$ with $0 < c < 1$. Then show that f is a contraction of $[a, b]$.

(7) Let X and Y be metric spaces. Assume that Y is a discrete metric space and that $f : X \longrightarrow Y$ is a contraction. What can you conclude about f?

(8) Define a sequence of positive real numbers by letting x_0 to be any positive real number and $x_{n+1} = (1 + x_n)^{-1}$. Show that this sequence converges and find its limit. (Hint: Prove that f is a contraction mapping where $f : [x_0, \infty) \to \mathbb{R}$ defined as $f(x) = \dfrac{1}{1+x}$).

Topology of Metric Spaces and Real Analysis: Practical 3.3
Connected Sets , Connected Metric Spaces
Objective Questions 3.3

. (Revised Syllabus 2018-19)

(1) Let (X, d) be a discrete metric space
(a) X is connected.
(b) X is connected only if X is infinite.
(c) X is connected if and only if X is a singleton set.
(d) None of above.

(2) Let d be usual distance in $\mathbb{R}$ and d_1 be the discrete metric in $\mathbb{R}$. Then
(a) $[0, 1]$ is a connected subset $(\mathbb{R}, d)$ as well as $(\mathbb{R}, d_1)$.
(b) $[0, 1]$ is connected subset of $(\mathbb{R}, d)$ but not connected subset of $(\mathbb{R}, d_1)$.
(c) $[0, 1]$ is not a connected subset of $(\mathbb{R}, d)$ but a connected subset of $(\mathbb{R}, d_1)$
(d) $[0, 1]$ is not a connected subset of $(\mathbb{R}, d)$ as well as $(\mathbb{R}, d_1)$

(3) If A is a connected subset of $(\mathbb{R}, d)$ (d being usual distance) then
(a) A° and $\overline{A}$ are connected.
(b) A° may not be connected but $\overline{A}$ is connected.
(c) Both A° and $\overline{A}$ may not be connected.
(d) A° is connected, but $\overline{A}$ may not be connected.

(4) Let A, B be connected subsets of $(\mathbb{R}, d)$ where d is the usual distance in $\mathbb{R}$. If $A \cap B \neq \emptyset$, then the following set may not be connected.
(a) $A \cup B$ (b) $A \cap B$ (c) $A \setminus B$ (d) $A \times B$ in $\mathbb{R}^2$ (Euclidean distance).

(5) Let $A \subseteq \mathbb{Q}$. If A is a connected subset of $(\mathbb{R}, d)$ where d is usual distance then
(a) $A = \mathbb{Q}$ (b) A is an infinite bounded set.
(c) A is a singleton set. (d) None of the above.

(6) Consider the following subsets of $(\mathbb{R}^2, d)$ where d Euclidean .
$(i)\{(x, y) \in \mathbb{R}^2 : xy = 1\}$ $(ii)\{(x, y) \in \mathbb{R}^2 : x = 0\}$
$(iii)\{(x, y) \in \mathbb{R}^2 : xy = 0\}$ Then,
(a) $(i), (ii)(iii)$ are all connected. (b) $(ii), (iii)$ are connected.
(c) Only (iii) is connected. (d) Only (i) is connected.

(7) Let A, B be non-empty closed subsets of a metric space (X, d). If $A \cup B$ and $A \cap B$ are connected subsets of X, Then6 (a) A and B are both connected.
(b) A and B are both not connected.
(c) A and B are connected if and only if $A = B$
(d) None of these.

(8) Let (X, d) be a finite metric space. If $A \subseteq X$ is connected then
(a) $A = X$ (b) $A \neq X$ (c) A is a singleton set. (d) A has more than one element.

(9) If A, B are connected subsets of $(\mathbb{R}^2, d)$ where d is usual distance and $A \cap B \neq \emptyset$, then
  (a) $A \cup B$ is connected but $A \cap B$ may not be connected.
  (b) $A \cup B$ may not be connected but $A \cap B$ is connected.
  (c) $A \cup B$ and $A \cap B$ are connected.
  (d) None of the above.

(10) Consider $(\mathbb{R}^2, d)$ where d is Euclidean metric and A be an open ball in $\mathbb{R}^2$ and L be a line in $\mathbb{R}^2$. Then
  (a) $A \cup L$ is connected if L does not intersect A.
  (b) $A \cup L$ is connected if L intersects A.
  (c) $A \cup L$ is disconnected if L intersects A but is not a tangent to A.
  (d) Cannot say.

(11) In $(\mathbb{R}^2, d)$ where d is Euclidean distance, the following set is not connected.
  (a) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$.
  (b) $\mathbb{R}^2 \setminus \{(0,0)\}$
  (c) $\mathbb{R}^2 \setminus \{(x,y) : y = 0\}$
  (d) None of the above.

(12) If A, B are connected subsets of $(\mathbb{R}, d)$ where d is usual and $A \cap B \neq \emptyset$, then
  (a) $A \cup B$ is connected but $A \cap B$ may not be connected.
  (b) $A \cup B$ may not be connected but $A \cap B$ is connected.
  (c) $A \cup B$ and $A \cap B$ are connected.
  (d) None of the above.

(13) Let A and B be connected subsets in a metric space (X, d) and $A \subseteq C \subseteq B$ Then,
  (a) C is connected .
  (b) C° is connected.
  (c) $\overline{C}$ is connected.
  (d) $C \cap \overline{A}$ is connected.

Topology of Metric Spaces and Real Analysis: Practical 3.3
Connected Sets , Connected Metric Spaces
Descriptive Questions 3.3

(1) Let (X, d) be a metric space and $A, B \subseteq X$ be closed. Prove that $A \cap B^c$ and $B \cap A^c$ separated.

(2) Let (X, d) be a metric space and $A, B, C \subseteq X$. If A and B are separated, B and C are separated, then prove that $A \cup C$ and B are separated.

(3) Find the components of the followings:
  (i) $[0,1] \cup [2,3]$ with usual distance.
  (ii) $(0,1) \cup \{2,3\}$ with usual distance .
  (iii) $\mathbb{Q}$

(iv) $\mathbb{R} \setminus \mathbb{Q}$
(v) $[0, 1]$ with distance metric.
(vi) $\{1, 2, 3\}$ with any metric.
(vii)$\mathbb{N}$ with usual distance .
(viii) $\{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$ with Euclidean distance in $\mathbb{R}^2$.

(4) Find the connected subsets of the following metric spaces:
(i) (X, d) where d is discrete metric.
(ii) (X, d) where X is a finite set.
(iii)$(\mathbb{N}, d)$ where d is usual distance in $\mathbb{R}$.
(iv) $(\mathbb{Q}, d)$ where d is usual distance in $\mathbb{R}$

(5) Show that the following subsets of $(\mathbb{R}^2, d)$ (d being Euclidean distance) are not connected.
(i) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = 1\}$
(ii) $\{(x, y) \in \mathbb{R}^2 : y \neq 0\}$
(iii) $\mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : y = 6\}$

(6) Prove or disprove:

(i) If A, C are connected subsets of a metric space and $A \subseteq B \subseteq C$, then B is connected.

(ii) If A° and ∂A are connected then A is connected.

(iii) If A, B are connected then $A \cup B, A \cap B$ are connected.

(iii) An open ball in a metric space is connected.

(iv) If A is a connected subset of a metric space (X, d), then A° and ∂A (boundary of A) are connected.

Topology of Metric Spaces and Real Analysis: Practical 3.4
Path Connectedness, Convex sets, Continuity and Connectedness
Objective Questions 3.4

. (Revised Syllabus 2018-19)

(1) Let (X, d) be a connected metric space. If $f : X \longrightarrow \mathbb{R}$ (d usual) is a non-constant continuous function .Then, $f(X)$ is
(a) finite set (b) countable set. (c) singleton set (d) uncountable set.

(2) The unit circle $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$ is (distance Euclidean)

(a) Compact and Connected
(b) Compact but not Connected
(c) Connected but not Compact
(d) neither Compact nor Connected

(3) Let (X, d) be a finite metric space ,$|X| \geq 2$. If $f : X \longrightarrow \mathbb{R}$ (usual distance) is a continuous function, then
(a) $|f(X)| \geq 2$
(b) $f(X)$ is connected.
(c) If $f(X)$ is connected then f is a constant function.
(d) None of these.

(4) Let A be a non-empty connected subset of $\mathbb{R}^2$ (distance Euclidean). Let $S = \{\|a\| : a \in A\}$. If every element in S is a rational number then
(a) A is a singleton set.
(b) Each point in A lies on a circle C_r where $C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$ for some $r \in \mathbb{Q}$
(c) Each point in A lies on a parabola $x^2 = ry$ for some $r > 0$.
(d) None of the above.

(5) Let (X, d) be a connected metric space and $A \subseteq X$ consider $\chi_A : X \longrightarrow \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

(a) If χ_A is continuous on X, then A is a finite set.
(b) If χ_A is continuous on X, then $A = \emptyset$ or $A = X$.
(c) If χ_A is continuous on X then A is a non-empty proper subset of X.
(d) None of the above.

(6) If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function, then $f([a, b])$ is
(a) $(0, M]$ for some $M > 0$ (b) (m, M) for some $m, M \in \mathbb{R}$
(c) $[m, M]$ for $m, M \in \mathbb{R}$ (d) None of these.

(7) Which of the following statements is false in $\mathbb{R}^n$?
(a) Continuous image of a compact set is compact.
(b) Continuous image of a connected set is connected.
(c) Continuous image of a path connected set is path connected.
(d) None of the above.

(8) Let (X, d) be a connected metric space which is not bounded. Let $x_0 \in X$ and $A_r = \{x \in X : d(x, x_0) = r\}(r > 0)$. Then
(a) $A_r = \emptyset$ except for finitely many positive real number r.
(b) $A_r \neq \emptyset \ \forall \ r > 0$
(c) $A_r = \emptyset \ \forall \ r > 0$
(d) None of these.

(9) Let (X, d) be a connected metric space and $f : X \longrightarrow \mathbb{Z}$ be a continuous map. Then
(a) f is onto . (b) f is one-one.
(c) f is bijective. (d) f is constant

(10) $\mathbb{R}^n \setminus \{0_{\mathbb{R}^n}\}$ is not path connected if
(a) $n = 3$, (b) $n = 4$ (c) $n = 1$ (d) None of these.

(11) In $(\mathbb{R}^2, d)$ (d Euclidean distance), the following set is not path connected.

(a) $\mathbb{R}^2 \setminus \mathbb{Q} \times \mathbb{Q}$ (c) $\mathbb{R}^2 - \{(x, y) : y = 0\}$
(b) $\mathbb{R}^2 \setminus \{(0, 0)\}$ (d) $B((0, 0), r) \setminus \{(0, 0)\}$

(12) In $(\mathbb{R}^2, d)$ (d Euclidean distance), the following set is path connected.
(a) $B((0, 0), 1) \cup \{(x, y) \in \mathbb{R}^2 : y = 1\}$
(b) $B((0, 0), 1) \cup \{(x, y) \in \mathbb{R}^2 : y = 2\}$
(c) $B((0, 0), 1) \cup \{(x, y) \in \mathbb{R}^2 : x = 2\}$
(d) None of the above.

(13) Which of the following statements is false:
(a) A path connected subset of $\mathbb{R}^n$ (distance being Euclidean) is connected.
(b) A connected subset of $\mathbb{R}^n$ (distance being Euclidean) is path connected.
(c) Union of two path connected subsets A, B in $\mathbb{R}^n$ distance being Euclidean such that $A \cap B \neq \emptyset$ is again path connected.
(d) If A, B are two path connected subsets of $\mathbb{R}^n$ (distance being Euclidean) such that $A \cap B \neq \emptyset$ then $A \cap B$ is path connected.

(14) Let (X, d) and (Y, d') be metric spaces. If $f : (X, d) \longrightarrow (Y, d')$ is a continuous function, then
(a) Number of components of $(X, d) \leq$ Number of components of (Y, d').
(b) Number of components of $(X, d) \geq$ Number of components of (Y, d').
(c) Number of components of $(X, d) =$ Number of components of (Y, d').
(d) Cannot say.

(15) Let (X, d) and (Y, d') be metric spaces and $f : (X, d) \longrightarrow (Y, d')$ be a bijective continuous function, then
  (a) Number of components of $(X, d) \leq$ Number of components of (Y, d').
  (b) Number of components of $(X, d) \geq$ Number of components of (Y, d').
  (c) Number of components of $(X, d) =$ Number of components of (Y, d').
  (d) Cannot say.

Topology of Metric Spaces and real Analysis: Practical 3.4
Path Connectedness, Convex sets, Continuity and Connectedness
Descriptive Questions 3.4

(1) Prove that the following subsets of $\mathbb{R}^n$ (distance being Euclidean) are convex and hence path connected. (i) an open ball (ii) a closed ball (iii) a line

(2) Let (X, d) be a metric space and A be a proper non-empty subset of X. If the characteristic function χ_A is continuous on X, show that X is not connected.

(3) Show that $B_r((0, 0)) \setminus \{(0, 0)\}$ is path connected in $\mathbb{R}^2$ with Euclidean distance.

(4) Show that $\mathbb{R}^2 \setminus S \times S$ where S is any countable subset of $\mathbb{R}$ is path connected. (Hint: For any $x, y \in \mathbb{R}^2 \setminus S \times S$ there are uncountable lines passing through x and y).

(5) Prove or disprove :

  (a) If A is a path connected subset of $\mathbb{R}^n$ (distance being Euclidean) thenA° is path connected.

  (b) If $\{A_n\}_{n\in\mathbb{N}}$ is a sequence of path connected subsets of $\mathbb{R}^2$ (distance being Euclidean) such that $A_{n+1} \subseteq A_n \quad \forall n \in \mathbb{N}$ and $\cap_{n\in\mathbb{N}} A_n \neq \emptyset$ then $\cap_{n\in\mathbb{N}} A_n$ is connected.

(6) (a) If (X, d) is a connected metric space and $f : X \longrightarrow \mathbb{Z}$ a continuous function, prove that f is constant.

  (b) If (X, d) is a connected metric space and (Y, d') is any metric space, Y being a finite set, then show that any continuous function $f : X \longrightarrow Y$ is constant.

  (c) Let (X, d) be a connected metric space and (Y, d_1) be a discrete metric space. Show that any continuous function $f : X \longrightarrow Y$ is constant.

(7) Let (X, d) be a connected metric space which is not bounded. Prove that for each $x_0 \in X$ and each $r > 0$, the set $\{x \in X : d(x, x_0) = r\}$ is non-empty.

(8) Give an example of a subset of $\mathbb{R}^n$ (distance being Euclidean) which is connected but not path connected.

(9) Show that if (X, d) is a connected metric space then either X is countable or X is singleton.

(10) Show that the following sets are path connected subsets of $\mathbb{R}^2$.

(i) $E = \{(x,y) \in \mathbb{R}^2, x > 0, x^2 - y^2 = 1\}$

(ii) $E_r = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}$

(iii) $E = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$

(iv) $E = \{(x,y) \in \mathbb{R}^2 : y^2 = x\} \cup \{(x,y) \in \mathbb{R}^2 : y^2 = -x\}$

(v) $E = \{(x,y) \in \mathbb{R}^2 : y = 0\}$

(vi) $E = \{(x,y) \in \mathbb{R}^2 : 1 < 2x + y < 3\}$

(vii) $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(viii) $E = \{(x,y) \in \mathbb{R}^2 : 0 < x < 2, 1 < y < 5\}$

Topology of Metric Spaces and Real Analysis: Practical 3.5
Pointwise and Uniform Convergence of Sequences of Functions and Properties
Objective questions 3.5

. (Revised Syllabus 2018-19)

(1) $\chi_n : \mathbb{R} \longrightarrow \mathbb{R}\ \chi_n(x) = \begin{cases} 1 & \text{if } x \in [-n, n] \\ 0 & \text{if } x \notin [-n, n] \end{cases}$

(a) $\{\chi_n\}$ converges pointwise to 0 on $\mathbb{R}$.
(b) $\{\chi_n\}$ does not converge uniformly on $\mathbb{R}$.
(c) $\{\chi_n\}$ converges uniformly to 1 on $\mathbb{R}$.
(d) None of the above.

(2) Let $f_n(x) = \sin nx$ for $x \in \mathbb{R}$. and $g_n(x) = \dfrac{f_n(x)}{n}\ \ \forall x \in \mathbb{R}$. Then

(a) $\{f_n\}$ and $\{_n\}$ are uniformly convergent on $\mathbb{R}$.
(b) $\{f_n\}$ and $\{_n\}$ are not pointwise convergent on $\mathbb{R}$.
(c) $\{g_n\}$ is uniformly convergent on $\mathbb{R}$ but $\{f_n\}$ is not.
(d) $\{f_n\}$ is uniformly convergent on $\mathbb{R}$ but $\{g_n\}$ is not.

(3) Let $f_n : [0, 1] \longrightarrow [0, 1]$ be defined by $f_n(x) = x * \chi_n(x)$ where $\chi_n(x) = \begin{cases} 0 & \text{if } x \notin \left[0, \dfrac{1}{n}\right] \\ 1 & \text{if } x \in \left[0, \dfrac{1}{n}\right] \end{cases}$

(a) $\{f_n\}$ converges uniformly to 0 on $[0, 1]$.
(b) $\{f_n\}$ converges pointwise to 1 on $[0, 1]$ but does not converge uniformly.
(c) $\{f_n\}$ converges uniformly to 1 on $[0, 1]$.
(d) None of the above.

(4) The least integer value of k for which $\left\{\dfrac{e^{-nx}}{n^k}\right\}$ is uniformly convergent on $[0, \infty)$ is

(a) 0 (b) 1 (c) -1 (d) 2

(5) If $\{f_n\}$ and $\{g_n\}$ are sequences of functions on $S, S \subseteq \mathbb{R}$ converging uniformly to f and g respectively on S then the following sequence of functions may not converge uniformly of S to the given function.

(a) $\{f_n + g_n\}$ to $f + g$. (b) $\{f_n - g_n\}$ to $f - g$. (c) $\{\lambda f_n\}$ to λf. (d) $\{f_n * g_n\}$ to $f * g$.

(6) Let $f_n(x) = \dfrac{x^n}{1 + x^n}, 0 \le x \le 1$.

(a) $\{f_n\}$ converges uniformly on $[0, 1]$
(b) $\{f_n\}$ converges uniformly on $\left[\dfrac{1}{2}, 1\right]$

(c) $\{f_n\}$ converges uniformly on $\left[0, \frac{1}{2}\right]$
(d) $\{f_n\}$ converges uniformly on $(0, 1]$

(7) Let $f_n(x) = \dfrac{x^n}{n} \quad \forall\, x \in [0, 1]$. Then
(a) $\{f_n\}$ converges uniformly to 0 but f_n' does not converge uniformly on $[0, 1]$.
(b) $\{f_n\}$ converges uniformly to 0 and f_n' converges uniformly to 1 on $[0, 1]$.
(c) $\{f_n\}$ does not converges uniformly on $[0, 1]$ but f_n' converges uniformly on $[0, 1]$.
(d) None of the above.

(8) Let $f_n(x) = \dfrac{x}{x+n}$ for $x \in [0, \infty)$. Show that $\{f_n\}$ does not converge uniformly on $[0, \infty)$ but converges uniformly on $[0, a]$ where $a > 0$. Also show that $\{f_n\}$ does not converge uniformly on $[a, \infty], a > 0$
(a) $\{f_n\}$ converges uniformly on $[0, \infty)$
(b) $\{f_n\}$ converges uniformly on $[a, \infty), a > 0$
(c) $\{f_n\}$ converges uniformly on $[0, a], a > 0$
(d) None of the above.

(9) $g_n(x) = x^{n-1}(1 - x), 0 \leq x \leq 1$.
(a) $\{g_n\}$ is uniformly convergent on $[0, 1]$.
(b) $\{g_n\}$ is not uniformly convergent on $[0, 1]$.
(c) $\{g_n\}$ is not pointwise convergent on $[0, 1]$.
(d) None of the above.

(10) $\{f_n\}$ and $\{g_n\}, g_n \neq 0$ are real valued functions on a non-empty subset $S, S \subseteq \mathbb{R}$ which are uniformly convergent to the functions f and g respectively on S.
(a) $\{fn * g_n\}$ need not be uniformly convergent on S.
(b) $\{f_n/g_n\}$ is uniformly convergent on S.
(c) $\{f_n * g_n\}$ is uniformly convergent to $f * g$ on S if each f_n is bounded on S.
(d) $\{f_n * g_n\}$ converges uniformly to $f * g$ on S if and only if either $f \equiv 0$ or $g \equiv 0$ on S.

(11) Let $\{f_n\}$ be a sequence of real valued functions on a set S converging uniformly to a function f. Then the following statement is **not** true.
(a) Each f_n is bounded on $S \Longrightarrow f$ is bounded on S.
(b) Each f_n is differential on $S \Longrightarrow f$ is differentiable on S
(c) Each f_n is continuous on $S \Longrightarrow f$ is continuous on S.
(d) Each f_n is integrable on $S \Longrightarrow f$ is integrable on S.

(12) Let $\{f_n\}$ be a sequence of real valued $R-$integrable functions on $[a, b]$ and f be the pointwise limit of $\{f_n\}$
(a) If $\lim\limits_{n \longrightarrow \infty} \displaystyle\int_a^b f_n \neq \int_a^b f$ then $\{f_n\}$ doesn't converge uniformly to f.
(b) If $\{f_n\}$ doesn't converge uniformly to f, then $\lim\limits_{n \longrightarrow \infty} \displaystyle\int_a^b f_n \neq \int_a^b f$.

(c) If $\lim\limits_{n\longrightarrow\infty}\displaystyle\int_a^b f_n \neq \int_a^b f$ then the convergence is uniform.
(d) None of the above.

(13) Let $\{f_n\}$ be a sequence of differentiable functions on (a, b). Let $\lim\limits_{n\longrightarrow\infty} f_n(x) = f(x)$, $\lim\limits_{n\longrightarrow\infty} f_n'(x) = g(x)$ (pointwise limits)
(a) If f is differntable on (a, b), then $f' = g$ on (a, b)
(b) If $\{f_n'\}$ converges uniformly to g, then f is differentable on (a, b) and $f' = g$.
(c) If $f' = g$ on (a, b) then $\{f_n\}$ converges uniformly to f on (a, b).
(d) If $\{f_n\}$ converges uniformly to f, then f is differentiable and $f' = g$ on (a, b)

(14) Let $f_n(x) = \begin{cases} x & \text{if } x \leq n \\ n & \text{if } x \geq n \end{cases}$
(a) $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a bounded function.
(b) $\{f_n\}$ converges uniformly on $\mathbb{R}$ to an unbounded functions.
(c) $\{f_n\}$ is not pointwise convergent on $\mathbb{R}$.
(d) $\{f_n\}$ converges pointwise on $\mathbb{R}$.

(15) Let $f_n(x) = \dfrac{x}{1 + nx^2}$
(a) $\{f_n\}$ converges uniformly on $\mathbb{R}$ but $\{f_n'\}$ does not converge uniformly on $\mathbb{R}$.
(b) $\{f_n\}$ converges uniformly on $\mathbb{R}$ and $\{f_n'\}$ also converges uniformly on $\mathbb{R}$.
(c) $\{f_n\}$ does not converge uniformly on $\mathbb{R}$ but $\{f_n'\}$ converges uniformly on $\mathbb{R}$.
(d) Neither $\{f_n\}$ nor $\{f_n'\}$ converge uniformly on $\mathbb{R}$.

(16) Let $f_n(x) = \dfrac{x^n}{1 + x^n}$ on $[0, 2]$ and $f(x) = \lim\limits_{n\longrightarrow\infty} f_n(x)$
(a) $\{f_n\}$ converges uniformly to f on $[0, 2]$ and f is continuous at $x = 1$.
(b) $\{f_n\}$ does not converge uniformly to f on $[0, 2]$ and f is not continuous on $[0, 1]$.
(c) $\{f_n\}$ does not converge uniformly to f on $[0, 2]$ but f is continuous on $[0, 2]$.
(d) None of the above.

(17) $f_n(x) = x^n$ for $x \in [0, 1]$
(a) The pointwise limit of $\{f_n\}$ is not continuous on $[0, 1]$
(b) $\{f_n\}$ converges pointwise on $[0, 1]$ to a continuous function.
(c) $\{f_n\}$ converges uniformly on $[0, 1]$ to a continuous function.
(d) None of the above.

(18) $f_n(x) = \dfrac{x^n}{n}$ for $x \in [0, 1]$. Let $\lim\limits_{n\longrightarrow\infty} f_n(x) = f(x)$, $\lim\limits_{n\longrightarrow\infty} f_n'(x) = g(x)$
(a) $\{f_n\}$ and $\{f_n'\}$ converge uniformly on $[0, 1]$.
(b) $\{f_n'\}$ converges uniformly to g on $[0, 1]$
(c) $\{f_n'\}$ does not converge uniformly to g on $[0, 1]$.
(d) None of the above

Topology of Metric Spaces and Real Analysis: Practical 3.5
Pointwise and Uniform Convergence of Sequences of Functions and Properties
DESCRIPTIVE QUESTIONS 3.5

(1) Show that each of the following sequences of functions converges pointwise on $(0,1)$. Identify the subintervals on which the convergence is uniform.

(i) $\dfrac{n}{nx+1}$ (ii) $\dfrac{x}{nx+1}$ (iii) $\dfrac{1}{nx+1}$

(2) Examine the following sequences of functions for pointwise and uniform convergence on $[0,1]$

(i) nxe^{-nx^2} (ii) $n^{\frac{1}{2}}x(1-x^2)^n$ (iii) $nx(1-x^2)^{n^2}$

(3) Examine the following sequences of functions for pointwise and the uniform convergence on $[0,\infty)$. In case of the convergence not being uniform, examine whether the convergence is uniform on $[0,a]$ or $[a,\infty)$ where $a>0$.

(i) e^{-nx} (ii) $\dfrac{\sin nx}{1+nx}$ (iii) x^2e^{-nx} (iv) $\dfrac{xe^{\frac{-x}{n}}}{n}$ (v) $n^2x^2e^{-nx}$

(4) $f_n:(0,\infty)\longrightarrow\mathbb{R}, f_n(x)=\dfrac{n}{1+nx}$. Then

(i) Show that $\{f_n\}$ is bounded on $(0,\infty)$ for each $n\in\mathbb{N}$.

(ii) Find the pointwise limit f of $\{f_n\}$ and show that f is not bounded on $(0,\infty)$.

(iii) Is $\{f_n\}$ uniformly convergent on $(0,\infty)$? State clearly the theorem you used.

(iv) Show that there does not exist $\alpha\in\mathbb{R}^+$ such that $|f_n(x)|\le\alpha$ for all $n\in\mathbb{N}$ and for all $x\in(0,\infty)$.

(5) Show that the following sequences of functions do not converge uniformly on the given domain.

(i) $f_n:[0,\infty)\longrightarrow\mathbb{R}, f_n(x)=\begin{cases} x & \text{if } x\le n\\ n & \text{if } x>n\end{cases}$

(ii) $f_n:[0,\infty), f_n(x)=\dfrac{nx}{1+nx^2}$.

(iii) $f_n:(0,1]\longrightarrow\mathbb{R}, f_n(x)=\begin{cases} 0 & \text{if } 0<x\le\frac{1}{n}\\ \dfrac{1}{x} & \text{if } \frac{1}{n}<x\le 1\end{cases}$

(6) $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = nxe^{-nx}$. Show that each f_n continuous on $[0,1]$, the pointwise limit of $\{f_n\}$ continuous on $[0,1]$ but $\{f_n\}$ does not converge uniformly convergent on $[0,1]$.

(7) Show that the following sequence of functions do not converge uniformly on the given domain.

$$f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = \begin{cases} nx & \text{if } 0 \leq x \leq \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}.$$

(8) Let $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = \dfrac{1}{nx+1}$.
Show that $\{f_n\}$ converges pointwise to f on $[a,b]$ and each f_n and f are $R-$integrable on $[0,1]$ with $\displaystyle\lim_{n\longrightarrow\infty}\int_a^b f_n(x)\ dx = \int_a^b f(x)dx$ but $\{f_n\}$ does converges uniformly on $[0,1]$.

(9) Let $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = \begin{cases} n^2 & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Show that $\{f_n\}$ does not converge uniformly on $[0,1]$. (Hint: show that if each f_n is R-integrable on $[0,1]$ and $f_n \longrightarrow f$ pointwise on $[0,1]$ but $\displaystyle\lim_{n\longrightarrow\infty}\int_a^b f_n(x)$ is not convergent.)

(10) Let $f_n : [0,1] \longrightarrow \mathbb{R}$ is defined for $n \geq 2$, $f_n(x) = \begin{cases} n^2x & \text{if } 0 \leq x \leq \frac{1}{n} \\ -n^2\left(x - \dfrac{2}{n}\right) & \text{if } \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & \text{if } \frac{2}{n} \leq x \leq 1 \end{cases}$. Show that $\{f_n\}$ does not converge uniformly on $[0,1]$.
(Hint: Show that each f_n is R-integrable on $[0,1]$ and $f_n \longrightarrow f$ pointwise on $[0,1]$, but $\displaystyle\lim_{n\longrightarrow\infty}\int_0^1 f_n(x) \neq \int_0^1 f(x)dx$.)

(11) Let $f_n : [-1,1] \longrightarrow \mathbb{R}, f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$. Given that $f_n \longrightarrow f$ uniformly on $[-1,1]$ where $f(x) = |x|$ for $x \in [-1,1]$. Find $\displaystyle\lim_{n\longrightarrow\infty}\int_{-1}^1 f_n(x)\ dx$.

(12) Let $f_n : [0,1] \longrightarrow \mathbb{R}, f_n(x) = x + n$. Does $\{f_n\}$ converge pointwise at any $x \in [-1.1]$. Does sequence $\{f_n\}$ converge uniformly on $[-1,1]$? Show that $\{f_n'\}$ converges uniformly on $[-1,1]$.

(13) $f_n : \mathbb{R} \longrightarrow \mathbb{R}, f_n(x) = \dfrac{e^{-n^2x^2}}{n}$. Find the pointwise limit function f of $\{f_n\}$ and g of $\{f_n'\}$. Does $f_n' \longrightarrow g$ uniformly on $\mathbb{R}$?. Is $f'(0) = g(0)$?

Topology of Metric Spaces and Real Analysis: Practical 3.6
Objective Questions 3.6

. (Revised Syllabus 2018-19)

(1) The series $\displaystyle\sum_{n=1}^{\infty}\frac{nx^2}{n^3+x^3}$ is
   (a) uniformly convergent on $[0, A]$ where $A > 0$ but not on $[0, \infty)$.
   (b) not uniformly convergent on $[0, A]$ where $A > 0$.
   (c) uniformly convergent on $[0, \infty)$.
   (d) none of the above.

(2) The series $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{n+1}$ is
   (a) uniformly convergent on $\mathbb{R}$.
   (b) not uniformly convergent on $[-a, a]$ where $0 < a < 1$
   (c) uniformly convergent on $[-a, a]$ where $0 < a < 1$.
   (d) none of the above.

(3) The series $\displaystyle\sum_{n=1}^{\infty}\frac{x^n}{x^n+1}$ is

   (a) pointwise convergent on $[1, \infty)$.
   (b) uniformly convergent on $[0, a], a < 1$.
   (c) uniformly convergent on $[0, \infty)$.
   (d) none of the above.

(4) The series $\displaystyle\sum_{n=1}^{\infty}\frac{x}{[(n-1)x+1)][nx+1]}$ is

   (a) uniformly convergent on $[0, \infty)$.
   (b) uniformly convergent on $[0, 1]$.
   (c) uniformly convergent on $[a, b], a > 0$.
   (d) none of the above.

(5) The series $\displaystyle\sum_{n=1}^{\infty}(-x)^n(1-x)$ is
   (a) uniformly convergent on $\mathbb{R}$.
   (b) uniformly on $[0, 1]$.
   (c) uniformly convergent on $[0, a]$ where $0 \le a < 1$ but not on $[0, 1]$.
   (d) none of the above.

(6) The least value of integer k for which $\displaystyle\sum_{n=1}^{\infty}\frac{\sin nx}{n^k}$ converges uniformly on $\mathbb{R}$ is
   (a) 1. (b) 2. (c) -1. (d) none of the above.

(7) $\sum_{n=1}^{\infty} |a_n|$ is convergent then $\sum_{n=1}^{\infty} a_n x^n$ is
(a) uniformly convergent on $\mathbb{R}$.
(b) uniformly convergent on any bounded interval.
(c) uniformly convergent on $[-a, a]$, where $0 \leq a < 1$.
(d) none of the above.

(8) The series $\sum_{n=1}^{\infty} x^n(1-x)$
(a) converges uniformly to x on $[0, a]$, where $0 \leq a < 1$.
(b) converges uniformly on $[0, 1)$.
(c) is not pointwise convergent at $x = 1$.
(d) none of the above.

(9) The series $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ (a) converges uniformly on $(0, \infty)$.
(b) converges uniformly on $[a, \infty), a > 0$.
(c) does not converge uniformly on $[a, \infty), a > 0$.
(d) none of the above.

(10) The series $\sum_{n=1}^{\infty} \frac{1}{(nx)^2}$
(a) converges uniformly on $\mathbb{R} \setminus \{0\}$.
(b) does not converge uniformly on $[a, \infty), a > 0$.
(c) converges uniformly on $[a, \infty), a > 0$.
(d) none of the above.

(11) Consider the series $\sum_{n=1}^{\infty} x^n(1-2x^n)$. Then
(a) $\int_0^1 \sum_{n=1}^{\infty} x^n(1-2x^n)dx \neq \sum_{n=1}^{\infty} \int_0^1 x^n(1-2x^n)dx$.
(b) $\int_0^1 \sum_{n=1}^{\infty} x^n(1-2x^n)dx = \sum_{n=1}^{\infty} \int_0^1 x^n(1-2x^n)dx$.
(c) it converges uniformly on $[0, 1]$ and can be integrated term by term.
(d) none of the above.

(12) Let $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$. Then
(a) None of the below statements are true.
(b) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is not uniformly convergent on $[0, 1]$ and cannot be integrated term by term.

(c) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is uniformly convergent on $[0,1]$ and can be integrated term by term.

(d) $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ is not uniformly convergent on $[0,\delta]$, where $0 \le \delta < 1$ and

$$\lim_{\delta \longrightarrow 1} \lim_{n \longrightarrow \infty} \int_0^{\delta} \sum_{k=1}^{n} \frac{\cos kx}{k^2} dx \neq \lim_{\delta \longrightarrow 1} \int_0^{\delta} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} dx$$

(13) The power series expansion for $\int_0^x e^{-t^2} dt$ is

(a) $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$ (b) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)}$ (c) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(n+1)!}$ (d) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n)!(n+1)}$

(14) If R is the radius of convergence of power series $\sum_{n=0}^{\infty} c_n x^n$, then radius of convergence of the power series $\sum_{n=0}^{\infty} c_n^k x^{nk}$ is (a) R^k (b) R (c) $R^{\frac{1}{k}}$ (d) $\frac{1}{R^k}$

(15) If R is the radius of convergence of power series $\sum_{n=0}^{\infty} c_n x^n$, then radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^{nk}$ is (a) R^k (b) R (c) $R^{\frac{1}{k}}$ (d) $\frac{1}{R^k}$

(16) If R is the radius of convergence of power series $\sum_{n=0}^{\infty} c_n x^n$ then the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2} c_n x^n$ is (a) R^2 (b) R (c) 0 (d) ∞

(17) $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R_1 and $\sum_{n=0}^{\infty} b_n x^n$, has radius of convergence R_2.

Let $C_n = \begin{cases} a_n & \text{if } n \text{ is even} \\ b_n & \text{if } n \text{ is odd} \end{cases}$. Then the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is

(a) $R_1 + R_2$ (b) $\min\{R_1, R_2\}$ (c) $\max\{R_1, R_2\}$ (d) None of the above.

(18) Let R be the radius of convergence of power series $\sum_{n=0}^{\infty} c_n x^n$, then the following powe series does not have radius of convergence $\mathbb{R}$.

(a) $\sum_{n=0}^{\infty} (-1)^n c_n x^n$ (b) $\sum_{n=0}^{\infty} \frac{c_n}{n} x^n$ (c) $\sum_{n=0}^{\infty} (-1)^n c_n^2 x^n$ (d) $\sum_{n=0}^{\infty} (-1)^n n c_n x^n$

(19) Let $\sum\limits_{n=0}^{\infty} c_n x^n$ be a power series with integer coefficients such that $c_n \neq 0$ for infinitely many n. If R is the radius of convergence of $\sum\limits_{n=0}^{\infty} x^n$, then

(a) $R = 0$ (b) $R = \infty$ (c) $R \leq 1$ (d) $R \geq 1$

(20) Let $f(x) = \sum\limits_{n=0}^{\infty} c_n x^n$ for $|x| < R$. If $f(x)$ is an even function for $|x| < R$, then

(a) $c_n = 0 \ \forall \ n \in \mathbb{N}$.
(b) $c_n = 0$ when n is even.
(c) $c_n = 0$ when n is odd.
(d) None of the above.

(21) If $\sum\limits_{n=0}^{\infty} c_n x^n$ has radius of convergence 1, then

(a) the power series converges at $x = 1$ and $x = -1$.
(b) the power series diverges at $x = 1$ and $x = -1$.
(c) the power series converges at $x = 1$ and diverges at $x = -1$.
(d) none of the above.

(22) Consider the power series $\sum\limits_{n=0}^{\infty} c_n x^n$, for which $c_n = \begin{cases} \dfrac{1}{2^k} & \text{if } n = 2k \\ 3^{k+1} & \text{if } n = 2k+1 \end{cases}$.

Then the radius of convergence of $\sum\limits_{n=0}^{\infty} c_n x^n$ is

(a) 2 (b) $\sqrt{2}$ (c) $\dfrac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

(23) If α is a non-zero real number then the radius of convergence of $\alpha^n x^n$ is

(a) $|\alpha|$ (b) $\dfrac{1}{|\alpha|}$ (c) 0 (d) ∞

(24) If α and β are real numbers such that $0 < |\beta| < |\alpha|$ then radius of convergence of $\sum\limits_{n=0}^{\infty} (\alpha^n + \beta^n) x^n$ is (a) $|\alpha|$ (b) $\dfrac{1}{|\alpha|}$ (c) $|\beta|$ (d) $\dfrac{1}{|\beta|}$

(25) The series expansion $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots$ is valid if

(a) $|x| \leq 1$ (b) $|x| \leq A$ for $A > 0$ (c) $|x| < 1$ (d) $x > 0$

(26) The series expansion $1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \dfrac{1}{(1-x)^2}$ is valid in

(a) $\mathbb{R}$ (b) $(-1, 1)$ (c) $[-1, 1)$ (d) $[a, b]$ for any $a, b \in \mathbb{R}, a < b$

(27) Let $E(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ for $x \in \mathbb{R}$. Then $\lim_{x \longrightarrow \infty} x^n E(-x) =$
(a) 1 (b) 0 (c) ∞ (d) -1

(28) Let $E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, C(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, x \in \mathbb{R}$. Then

(a) $E(x), C(x), S(x)$ are one-one
(b) Only $E(x)$ is one-one
(c) $C(x), S(x)$ are one-one
(d) None of the above.

(29) Let $L : (0, \infty) \longrightarrow \mathbb{R}$ be defined as $L(E(x)) = x$ and $E(L(y)) = y, x \in \mathbb{R}$.

(a) $L(1-y) = -\sum_{n=1}^{\infty} \frac{y^n}{n}$
(b) $L(y) = \sum_{n=1}^{\infty} \sum_{n=0}^{\infty} n y^n$
(c) $L(y) = \sum_{n=1}^{\infty} \frac{y^n}{n+1}$
(d) None of the above.

(30) Let $L : (0, \infty) \longrightarrow \mathbb{R}$ be defined as $L(E(x)) = x$ and $E(L(y)) = y, x \in \mathbb{R}$. Then
(a) L is represented by power series on $(0, 1)$. (b) L is represented by power series on $(0, \infty)$. (c) L is not represented by power series. (d) None of the above.

(31) Let $\cosh x = \frac{E(x) + E(-x)}{2}$ and $\sinh x \frac{E(x) - E(-x)}{2}, x \in \mathbb{R}$. Then the following identity is not true.
(a) $\sinh(-x) = -\sinh x, \cosh(-x) = -\cosh x$ (b) $\sinh(x + y) = \sinh x \cosh(y) + \cosh x \sinh y$ (c) $\frac{d}{dx} \sinh x = \cosh x$ (d) $\cosh^x + \sinh^2 x = 1$

Topology of Metric Spaces and Real Analysis: Practical 3.6
Descriptive Questions 3.6

(1) Show that $\sum_{n=1}^{\infty} x^n(1-x)$ converges uniformly to x on $[0, a]$, where $0 \leq a < 1$, but $\sum_{n=1}^{\infty} x^n(1-x)$ is not uniformly convergent on $[0, 1)$.

(2) Show that the series $\sum_{n=1}^{\infty} (-x)^n(1-x)$ converges uniformly on $[0, 1]$.

(3) (i) Show that $\sum_{n=1}^{\infty} \frac{1}{(nx)^2}$ does not converge uniformly on $\mathbb{R} \setminus \{0\}$ but converges uniformly on $[a, \infty), a > 0$.

(ii) Show that $\sum_{n=1}^{\infty} \frac{1}{x^2+n^2}$ is uniformly convergent on $\mathbb{R}$.

(iii) Show that $\sum_{n=1}^{\infty} \frac{1}{x^n+1}$ is uniformly convergent on $[a,\infty), a>1$.

(iv) Show that $\sum_{n=1}^{\infty} \frac{x^n}{x^n+1}$ is uniformly convergent on $[0,a], a<1$ but not pointwise convergent on $[1,\infty)$.

(v) Show that $\sum_{n=1}^{\infty} \frac{x^2}{(1+x^2)^n}$ does not converge uniformly on $(0,\infty)$ but converges uniformly on $[a,\infty), a>0$.

(4) Show that each of the following series of functions converges uniformly on the indicated interval.

(i) $\sum_{n=1}^{\infty} e^{-nx}x^n, [0,A], A>0$.

(ii) $\sum_{n=1}^{\infty} \frac{e^{-nx}}{n}, x\in[a,\infty), a>0$.

(iii) $\sum_{n=1}^{\infty} e^{-nx}$ on $[a,\infty), a>0$.

(5) If $\sum_{n=1}^{\infty} |a_n|<\infty$, then the series $\sum_{n=1}^{\infty} a_n \cos nx$ and $\sum_{n=1}^{\infty} a_n \sin nx$ converge on $\mathbb{R}$.

(6) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n(x^2+n)}{n^2}$ converges uniformly every bounded subset of $\mathbb{R}$.

(7) Show that $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}, p\leq 1$ is uniformly convergent on $S=[-\pi,-a]\cup[a,\pi], a>0$.

(8) (i) $\sum_{n=1}^{\infty} 2x\left[\frac{e^{\frac{-x^2}{n^2}}}{n^2}-\frac{e^{\frac{-x^2}{(n+1)^2}}}{(n+1)^2}\right]$ in $[a,b]$. Show that the series converges uniformly to $2xe^{-x^2}$ on $[a,b]$. Hence show that $\sum_{n=1}^{\infty}\int_a^b 2x\left[\frac{e^{\frac{-x^2}{n^2}}}{n^2}-\frac{e^{\frac{-x^2}{(n+1)^2}}}{(n+1)^2}\right]dx=e^{-a}-e^{-b}$.

(ii) $\sum_{n=1}^{\infty} x^n(1-2x^n)$. Show that the series does not converge pointwise at $x = 1$ but converges pointwise to $\frac{x}{1+x}$ on $[0,1)$. Show that $\int_0^1 \frac{x}{1+x}dx \neq \sum_{n=1}^{\infty}\int_0^1 x^n(1-2x^n)dx$. Hence show that the series does not converge uniformly on $[0,1)$. State the result you used . (Let D be a bounded subset of $\mathbb{R}$ and let $f : D \longrightarrow \mathbb{R}$ be a function. We say that f is integrable over D if f is a bounded function and if there are $a, b \in \mathbb{R}$ with $D \subseteq [a, b]$ such that the function $f^* : [a, b] \longrightarrow \mathbb{R}$ defined by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

is integrable on $[a, b]$. In this case, the Riemann integral of f over D is defined by

$$\int_D f(x)dx = \int_a^b f^*(x)dx.$$

Reference: A Course in Calculus an Real Ananlysis, Sudhir R. Ghorpade, Balmohan V. Limaye,Second Edition, Springer, pg. no. 216)

(iii) $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ in $[0,1]$.

Show that $\int_0^1 \left[\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]\right] dx = \sum_{n=1}^{\infty}\int_0^1 \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right] dx.$

but $\sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$ does not converge uniformly on $[0,1]$.

(9) Show that $\sum_{n=1}^{\infty}\frac{1}{n^3+n+x^2}$ is uniformly convergent on $\mathbb{R}$ and check that it can be differentiated term by term.

(10) Find the radius of convergence of each of the following power series.

(i) $\sum_{n=0}^{\infty} n^3x^n$

(ii) $\sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$

(iii) $\sum_{n=0}^{\infty}\frac{n^3}{3^n}x^n$

(iv) $\sum_{n=0}^{\infty}(n^3-5n^2+7n-2)x^n$

(v) $\sum_{n=0}^{\infty}\frac{e^n}{n+1}x^n$

(vi) $\sum_{n=0}^{\infty}\frac{x^n}{(n+1)^{\sqrt{n}}}$

(11) Find the interval of convergence of the following power series.

(i) $\sum_{n=0}^{\infty}\frac{(x-1)^{n-1}}{3^nn^2}$

(ii) $\sum_{n=0}^{\infty}\frac{n!(x-2)^n}{n^n}$

(iii) $\sum_{n=0}^{\infty}\frac{(x^2-1)^n}{2^n}$

(iv) $\displaystyle\sum_{n=0}^{\infty}\frac{(3x+6)^n}{n!}$ (v) $\displaystyle\sum_{n=0}^{\infty}\frac{(x+3)^{n-1}}{n}$

(12) Find the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$, where $c_n = \dfrac{h(h-1)\cdots(h-n+1)}{n!}$.

(13) Consider the power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ with integer coefficients. If $c_n \neq 0$ for infinitely many n, then show that its radius of convergence is at most 1.

(14) Give an example of a power series with radius of convergence $= 5$ and interval of convergence $= (3, 13)$.

(15) If $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is a power series such that $0 < \alpha < |c_n| < \beta \quad \forall\, n \in \mathbb{N}$ where $\alpha, \beta \in \mathbb{R}$, find its radius of convergence.

(16) Let $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $\displaystyle\sum_{n=0}^{\infty} b_n x^n$ be power series such that

$$a_n = \begin{cases} 1 & \text{if } n \text{ is square of an integer} \\ 0 & \text{otherwise} \end{cases} \qquad b_n = \begin{cases} 1 & \text{if } n = k! \text{ for some } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ and $\displaystyle\sum_{n=1}^{\infty} b_n x^n$.

(17) If $0 < |\alpha| < |\beta|$ then find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty}(\alpha^n + \beta^n)x^n$ and $\displaystyle\sum_{n=0}^{\infty}(\alpha^n + \beta^n)x^n$.

(18) Show that $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n$ for $|x| < 1$.

(19) By differentiating a suitable power series term by term, obtain the formula,

$$1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots = \frac{1}{(1-x)^2}$$

for $-a \leq x \leq a$. What should be the value of $'a'$ so that term by term differentiation is valid?

(20) If $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$ for $x \in \mathbb{R}$ and $\frac{d}{dx}(\sin x) = \cos x, \forall x \in \mathbb{R}$, then show that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$

(21) Show by integrating the series for $\frac{1}{1+x}$, that $\log(1+x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}$.

(22) By integrating a suitable powe series over an interal $[0,t]$, where $0 \leq t \leq 1$, show that $\frac{1}{2} = \sum_{n=1}^{\infty} \frac{1}{n!(n+2)}$.

(23) For $|x| < 1$, show that $\sin^{-1} x = \sum_{n=0}^{\infty} \frac{1.3.5.\cdots.(2n-1)x^{2n+1}}{2.4.\cdots.(2n)(2n+1)}$.

(24) For $|x| < 1$, show that $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$.

(25) Find a series expansion for $\int_0^x e^{-t^2} dt$ for $x \in \mathbb{R}$.

(26) If $\sum_{n=0}^{\infty} |a_n| < \infty$, prove that $\int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n\right) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$.

Topology of Metric Spaces and Real Analysis: Practical 3.7 Miscellaneous.

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Revised Syllabus 2018-19

UNIT I : Continuous functions on Metric Spaces

(1) Let (X,d) and (Y,d') be metric spaces. Show that $f : X \to Y$ is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.

(2) Let (X,d) and (Y,d') be metric spaces and $f : X \longrightarrow Y$. Show that the following statements are equivalent.

   (i) f is continuous on X.

   (ii) For each open subset G of $Y, f^{-1}(G)$ is an open subset of X.

   (iii) For each closed subset F of $Y, f^{-1}(F)$ is a closed subset of X.

(3) Let (X,d) and (Y,d') be metric spaces. Show that $f : X \longrightarrow Y$ is continuous at $p \in X$ if and only if for each sequence (x_n) in X converging to p, the sequence $(f(x_n))$ converges to $f(p)$ in Y.

(4) Let (X,d) and (Y,d') be metric spaces. Show that f is continuous at $x \in X$ if and only if for each open subset U of Y containing $f(x), \exists$ an open subset V of X containing x such that $f(V) \subseteq U$.

(5) Let (X,d) and $(Y,d'), (Z,d'')$ be metric spaces. If $f : X \longrightarrow Y$ is continuous and $g : Y \longrightarrow Z$ is continuous, then show using $\epsilon - \delta$ definition or sequential criterion that $g \circ f : X \longrightarrow Z$ is continuous. Give an example to show that converse of the above statement is not true.

(6) Let (X,d) and (Y,d') be metric spaces. Show that $f : X \longrightarrow Y$ is continuous on X if and only if for each subset A of $X, f(\overline{A}) \subseteq \overline{(f(A))}$.

(7) Let (X,d) and (Y,d') be metric spaces. Show that $f : X ßY$ is continuous on X if and only if for each subset B of $Y, \overline{(f^{-1}(B))} \subseteq f^{-1}(\overline{B})$.

(8) Let (X,d) and (Y,d') be metric space and $f : X \longrightarrow R$ (usual distance) be a continuous function. If $f(x_0) > 0$ for some $x_0 \in X$, show that $f(x) > 0, \forall x \in B(x_0, \delta)$.

(9) Let (X,d) and (Y,d') be metric spaces. When is $f : X \longrightarrow Y$ said to be uniformly continuous? Give an example to show that a continuous map need not be uniformly continuous.

(10) Let (X,d) and (Y,d') be metric spaces. If $f, g : X \longrightarrow Y$ are continuous functions, then show that $F = \{x \in X : f(x) = g(x)\}$ is a closed subset of X. Hence, deduce that if $f(x) = g(x), \forall x \in D$, where D is a dense subset of X, then $f = g$.

(11) Let (X, d) and (Y, d') be metric spaces. Show that $f : (X, d) \longrightarrow (Y, d')$ is a continuous function if and only if $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$ for each subset B of Y.

(12) Let (X, d) be a metric space and $A \subseteq X$. Using $\epsilon - \delta$ definition show that $f_A(x) = d(x, A)$ is a continuous map from (X, d) to $(\mathbb{R}, d)$ where d is the usual distance on $\mathbb{R}$.

(13) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function (distance is Euclidean) and F be a closed subset of $\mathbb{R}$. Show that $A = \{x \in F : f(x) = 0\}$ is a closed set in $\mathbb{R}$. Is the result true if F is not closed?

(14) Let (X, d) be a metric space. Show that $f : (X, d) \longrightarrow (\mathbb{R}, d)$ (where d is usual distance) is continuous if and only if $f^{-1}(-\infty, a)$ and $f^{-1}(a, \infty)$ are both open in (X, d) for each $a \in \mathbb{R}$.

(15) Show that the metrics d and d_1 on a set X are equivalent if and only if $i : (X, d) \longrightarrow (X, d_1)$ and $i : (X, d_1) \longrightarrow (X, d)$ are continuous functions, where i denotes the identity map on X.

(16) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ (with respect to usual distance) and $A = \{(x, y) : y < f(x)\}, B = \{(x, y) : y > f(x)\}$. Show that f is continuous on $\mathbb{R}$ if and only if A, B are open subsets of $(\mathbb{R}^2, d)$ where d is the Euclidean distance.

(17) Let X be a finite set and d be any metric on X. Show that any function $f : X \longrightarrow Y$ is continuous, where (Y, d') is a metric space.

(18) Let (X, d) be a discrete metric space and (Y, d') be any metric space. Show that any function $f : X \longrightarrow Y$ is continuous.

(19) Show that any function $f : (\mathbb{N}, d) \longrightarrow (X, d')$ is continuous, where d is usual distance on $\mathbb{N}$ and (X, d') is any metric space.

(20) Show that any function $f : (\mathbb{Z}, d) \longrightarrow (X, d')$ is continuous, where d is usual distance on $\mathbb{Z}$ and (X, d') is any metric space.

(21) (X, d) and (Y, d') are metric space and $f : X \longrightarrow Y$ is continuous. Give examples to show that

  (i) G is an open subset of X does not imply $f(G)$ is an open subset of Y.
  (ii) F is a closed subset of X does not imply $f(F)$ is a closed subset of Y.
  (iii) (x_n) is a Cauchy sequence in X does not imply the sequence $(f(x_n))$ is a Cauchy in Y.

(22) Let (X, d) be a metric space and (Y, d') be any metric spaces. If $f : (X, d) \longrightarrow (Y, d')$ is a continuous function, then show that $f(X)$ is a compact set.

(23) Let (X, d) and (Y, d') be metric spaces and $f : X \longrightarrow Y$ be continuous. If (X, d) is a compact metric space, then show that $f : X \longrightarrow Y$ is uniformly continuous.

(24) Let (X, d) be a complete metric space. $T : X \longrightarrow X$ be a contraction map. Then show that T has a fixed point.

(25) Let (X, d) be a complete metric space and $T : X \longrightarrow X$ be a mapping such that $T^m = T \circ T \circ T \circ \ldots \circ T(m\text{times})$ is a contraction for some fixed m then show T has an unique fixed point.

(26) Let (X, d) be a compact metric space and $T : X \longrightarrow X$ be such that $d(T(x), T(y)) < d(x, y)$ then show that T has unique fixed point in X.

UNIT II : Connected sets

(1) Let (X, d) be a metric space. Prove that the following statements are equivalent:

  (i) X can be expressed as a union of two non-empty separated sets.
  (ii) X can be expressed as a union of two non-empty disjoint closed sets.
  (iii) X can be expressed as a union of two non-empty disjoint open sets.
  (iv) There is a non-empty proper subset of X which is both open and closed.

(2) Show that A is a connected subset of $\mathbb{R}$ with respect to the usual distance if and only if it is an interval.

(3) Let (X, d) be a connected metric space and (Y, d') be any metric space. If $f : (X, d) \longrightarrow (Y, d')$ is a continuous function, then show $f(X)$ is a connected set.

(4) Show that a metric space (X, d) is connected if and only if every continuous function $f : X \longrightarrow \{1, -1\}$ is constant.

(5) If a metric space (X, d) is connected and A is a non-empty proper subset of X, then show that δA, boundary of A is non-empty.

(6) Show that a metric space (X, d) is connected if and only if for each $a, b \in X$, there is a connected subset E of X such that $a, b \in E$.

(7) Let (X, d) be a metric space. If A is a connected subset of X, and $A \subseteq B \subseteq \overline{A}$ then show that B is connected. Hence, show that $\overline{A}$ is connected. Give an example to show that if A, C are connected subset of X and $A \subseteq B \subseteq C$ then B need not be connected.

(8) If A and B are connected subset of a metric space (X, d), and $A \cap B \neq \emptyset$, then show that $A \cup B$ is connected. Give an example to show that $A \cap B$ need not be connected.

(9) Let (X, d) be a metric space. If $\{A_\alpha : \alpha \in \Lambda\}$ is a family of connected subsets of X such that $\cap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$, then show that $\cup_{\alpha \in \Lambda} A_\alpha$ is connected.

(10) Let (X, d) be a metric space. If $\{A_n : n \in \mathbb{N}\}$ is a family of connected subsets of X such that $A_n \cap A_{n+1} \neq \emptyset$ for each $n \in \mathbb{N}$, then show that $\cup_{n \in \mathbb{N}} A_n$ is connected.

(11) Prove that an open ball in $\mathbb{R}^n$ is a convex set. (The distance being Euclidean). Hence, deduce that it is path connected.

(12) Show that a path connected subset of $\mathbb{R}^n$ is connected.

(13) Let A and B be path connected subsets of a metric space (X, d) such that $A \cap B \neq \emptyset$. Show that $A \cup B$ is path connected.

(14) Let (X, d) and (Y, d') be metric spaces. If (X, d) is path connected and $f : X \longrightarrow Y$ is continuous, show that $f(X)$ is path connected.

(15) Let (X, d) be a metric space and A be a non-empty subset of X.
Prove or disprove: IfA is connected, then A°, and ∂A are connected. Give an example to show that A° and ∂A may be connected, but A may not be connected.

(16) Let (X, d) be a metric space. If A is connected subset of X, then show that $\overline{A}$ is connected. Give an example to show that A° may not be connected.

UNIT III : Sequences and series of functions

(1) **Mn Test**: A sequence $\{f_n\}$ of real valued functions on S ($S \subseteq \mathbb{R}$) converges uniformly to a function $f : S \longrightarrow \mathbb{R}$ on S if and only if, $\lim\limits_{n \longrightarrow \infty} M_n = 0$ where
$M_n = \sup\{|f_n(x) - f(x)| : x \in S\}$. Hence show that if there is a sequence (t_n) in $\mathbb{R}$ such that $|f_n(x) - f(x)| \leq t_n$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$ and for all $x \in S$ such that $t_n \longrightarrow 0$, then $f_n \longrightarrow f$ uniformly on S.

(2) State and prove Cauchy Criterion for uniform convergence of sequences of functions.

(3) Let $\{f_n\}$ be a sequence of real valued functions defined on a set $S \subseteq \mathbb{R}$ such that $f_n \longrightarrow f$ uniformly on S. If each f_n is bounded on S, then prove the following.

   (i) f is bounded on S.
   (ii) there exists $\alpha \in \mathbb{R}^+$ such that $|f_n(x)| \leq \alpha$ for all $n \in \mathbb{N}$ and for all $x \in S$.
   (iii) $\sup\{f_n(x) : x \in S\} \longrightarrow \sup\{f(x) : x \in S\}$.
   (iv) $\inf\{f_n(x) : x \in S\} \longrightarrow \inf\{f(x) : x \in S\}$.

(4) Let $\{f_n\}$ be a sequence of real valued continuous functions defined on a subset S of $\mathbb{R}$ such that $f_n \longrightarrow f$ uniformly on S. Then prove the following.

   (i) f is continuous on S.
   (ii) For any $p \in S$, $\lim\limits_{n \longrightarrow \infty} \lim\limits_{x \longrightarrow p} f_n(x) = \lim\limits_{x \longrightarrow p} \lim\limits_{n \longrightarrow \infty} f_n(x)$.

(5) Let $\{f_n\}$ be a sequence of real valued $R-$integrable functions defined on $[a, b]$ such that $f_n \longrightarrow f$ uniformly on [a, b] . Then prove that f is $R-$ integrable on $[a, b]$ and $\lim\limits_{n\longrightarrow\infty} \int_a^b f_n(t)\, dt = \int_a^b \lim\limits_{n\longrightarrow\infty} f_n(t)\, dt.$

(6) $\{f_n\}$ is a sequence of real valued $R-$integrable functions on $[a, b]$ converging uniformly to f on $[a, b]$. If $F_n(x) = \int_a^x f_n(t)\, dt$ then prove that $\{F_n\}$ converges uniformly to F on $[a, b]$ where $F(x) = \int_a^x f(t)dt.$

(7) Let $\{f_n\}$ and $\{g_n\}$ be sequences of real valued bounded functions on S subset of $\mathbb{R}$. If $\{f_n\}$ and $\{g_n\}$ converge uniformly to f and g respectively on S, then prove that $\{f_n * g_n\}$ is uniformly convergent on S.

(8) Let $\{f_n\}$ be a sequence of real valued continuously differentiable functions on $[a, b], a < b$ such that $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a, b]$ and $\{f_n'\}$ converges uniformly on $[a, b]$. Then

  (i) there is a continuously differentiable function f on $[a, b]$ such that $f_n \longrightarrow f$ uniformly on $[a, b]$ and
  (ii) $f_n' \longrightarrow f'$ uniformly on $[a, b]$.

(9) Let $\{f_n\}$ be a sequence of differentiable real valued functions on a bounded interval I. If $\{f_n(x_0)\}$ is convergent for some $x_0 \in I$ and $\{f_n'\}$ converges uniformly to g on I then $\{f_n\}$ converges uniformly on I and if $\{f_n\}$ converges uniformly to f on I then f is differentiable on I and $f' = g$ on I.

(10) State and prove Cauchy Criterion for Uniform Convergence of a Series $\sum\limits_{n=0}^{\infty} f_n$ of real valued functions on a subset S of $\mathbb{R}$.

(11) State and prove Weierstrass M-Test for the convergence of a series $\sum\limits_{n=1}^{\infty} f_n$ of real valued functions defined on subset S of $\mathbb{R}$.

(12) Let $\{f_n\}$ be a sequence of real-valued bounded functions on a set $S \subseteq \mathbb{R}$. If the series $\sum\limits_{n=1}^{\infty} f_n$ converges uniformly to the sum function f on S then prove that f is also bounded on S.

(13) If $\{f_n\}$ is a sequence if real valued continuous functions on $S, S \subseteq \mathbb{R}$ such that $\sum_{n=1}^{\infty} f_n$ converges uniformly to f on S, then prove that f is continuous on S, and for $p \in S$, $\sum_{n=1}^{\infty} \lim_{x \longrightarrow p} f_n(x) = \lim_{x \longrightarrow p} \sum_{n=1}^{\infty} f_n(x)$.

(14) Let $\sum_{n=1}^{\infty} f_n$ be a series of $R-$integrable functions on $[a, b]$, converging uniformly to f on $[a, b]$, then prove that f is $R-$integrable on $[a, b]$ and $\int_a^b f(x)\ dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\ dx$.

(15) If $\{f_n\}$ is a sequence of differentiable functions on $[a, b]$ such that each f_n' is continuous on $[a, b]$ and if $\sum_{n=1}^{\infty} f_n$ converges to f pointwise on $[a, b]$ and $\sum_{n=1}^{\infty} f_n'$ converges uniformly on $[a, b]$ then prove that $f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad$ for $a \leq x \leq b$.
NOTE: For Q. No. (12) to (15), the corresponding result about uniform convergence of sequence of functions can be used directly.

(16) If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges at $x_1 \in \mathbb{R}, x_1 \neq 0$ and diverges at $x_2 \in \mathbb{R}$ then the power series $\sum_{n=0}^{\infty} |c_n x^n|$ converges for all $x \in \mathbb{R}$ with $|x| < |x_1|$ and diverges for all $x \in \mathbb{R}$ with $|x| > |x_2|$.

(17) A power series $\sum_{n=1}^{\infty} c_n x^n$ is either absolutely convergent for all $x \in \mathbb{R}$, or there is a unique real number $r \geq 0$ such that the series is absolutely convergent for each $x \in \mathbb{R}$ with $|x| < r$ and is divergent for each $x \in \mathbb{R}$ with $|x| > r$.

(18) Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series with coefficients in $\mathbb{R}$. Let $\alpha = \limsup_{n \longrightarrow \infty} \left| c_n \right|^{\frac{1}{n}}$. Then the radius of convergence r of $\sum_{n=0}^{\infty} c_n x^n$ is $\frac{1}{\alpha}$ (if $\alpha = 0, r = \infty$ and if $\alpha = \infty, r = 0$) (Statement Only).

Definition: **limit superior of a sequence** $\left(\limsup\limits_{n\longrightarrow\infty} a_n\right)$: Let (a_n) be a sequence in $\mathbb{R}$.

i. If (a_n) is not bounded above then $\limsup\limits_{n\longrightarrow\infty} a_n = \infty$

ii. If (a_n) is bounded above then for each $n \in \mathbb{N}$, define, $M_n = \sup\{a_k : k \geq n\}$. Then sequence (M_n) is monotonic decreasing. If sequence (M_n) is bounded below then it is convergent. In such case, $\limsup\limits_{n\longrightarrow\infty} a_n = \lim\limits_{n\longrightarrow\infty} M_n$.

iii. If sequence (M_n) is not bounded below then $\limsup\limits_{n\longrightarrow\infty} a_n = -\infty$.

It can be proved that if sequence (a_n) is convergent then $\limsup\limits_{n\longrightarrow\infty} a_n = \lim\limits_{n\longrightarrow\infty} a_n$

(19) Let $\sum\limits_{n=0}^{\infty} c_n x^n$ be a power series with coefficients in $\mathbb{R}$ and there exist $n_0 \in \mathbb{N}$ such that $c_n \neq 0$, $\forall n \geq n_0$. Let $\alpha = \lim\limits_{n\longrightarrow\infty} \left|\frac{c_{n+1}}{c_n}\right|$. Then the radius of convergence r of $\sum\limits_{n=0}^{\infty} c_n x^n$ is $\frac{1}{\alpha}$ (if $\alpha = 0, r = \infty$ and if $\alpha = \infty, r = 0$) (Statement Only).

(20) Let r be the radius of convergence of a power series $\sum\limits_{n=1}^{\infty} c_n x^n$. If $s \in \mathbb{R}$ is such that $0 < s < r$, then prove that the power series converges uniformly on $[-s, s]$. Further, let $f : (-r, r) \longrightarrow \mathbb{R}$ be the sum function of the power series $\sum\limits_{n=0}^{\infty} c_n x^n$ then prove that

(i) f is continuous on $(-r, r)$.

(ii) For every $x \in (-r, r)$, $\int_0^x f(t)dt = \sum\limits_{n=0}^{\infty} c_n \frac{x^{n+1}}{n+1}$

(iii) f is differentiable on $(-r, r)$ and $f'(x) = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$ for $x \in (-r, r)$.

(iv) f is infinitely differentiable on $(-r, r)$, and $c_n = \frac{f^{(n)}(0)}{n!}$ for $n \in \mathbb{N}, c_0 = f(0)$.