Practical -03
Objectives

Practical no 3. Ring, Subring, Ideal and Integral domain

1. Let R be a ring and a, b be non-zero elements of R. The equation $ax = b$
   (a) has a unique solution in R. ✓
   (b) may have more than one solution in R.
   (c) has atmost one solution in R.
   (d) None of these.

R be a ring a, b be non zero-element
If R is having units then for every non zero element a, ∃ $a^{-1} \in R$ s.t $aa^{-1} = 1 = a^{-1}a$

$\therefore \quad ax = b \Rightarrow a^{-1}(ax) = a^{-1}b$
$\Rightarrow (a^{-1}a)x = a^{-1}b$
$\Rightarrow x = a^{-1}b$

2. The group of units of the ring $\mathbb{Z}_{25}$ is
   (a) $\{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ mod 25.
   (b) $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{11}, \bar{12}, \bar{13}, \bar{14}, \bar{16}, \bar{17}, \bar{18}, \bar{19}, \bar{21}, \bar{22}, \bar{23}, \bar{24}\}$ mod 25. ✓
   (c) $\{\bar{1}, \bar{4}, \bar{8}, \bar{12}, \bar{16}, \bar{20}\}$ mod 25.
   (d) $\{\bar{1}, \bar{3}, \bar{6}, \bar{9}, \bar{12}, \bar{15}, \bar{18}, \bar{21}, \bar{24}\}$ mod 25.

a element in R is said to be on unit if
$a.b = 1$ 101

$Z_{25} = \{\bar{0}, \bar{1}, \bar{2} \ldots \bar{24}\}$ mod 25 126

$3 \times 17 = 51 \equiv 1 \pmod{25} \Rightarrow 3$ & 17 are units
$2 \times 13 = 26 \equiv 1 \pmod{25} \Rightarrow 2$ & 13 " "
$4 \times 19 = 76 \equiv 1 \pmod{25} \Rightarrow 4$ & 19 " "
$6 \times 21 = 126 \equiv 1 \pmod{25} \Rightarrow 6$ & 21 " "
$7 \times 18 = 126 \equiv 1 \pmod{25} \Rightarrow 7$ & 18 " "

3. The group of units of a ring is
   (a) abelian but may not be cyclic ✓ (b) Cyclic (c) may not be abelian (d) finite

$a.b = 1 = b.a$
$G = \{a, b \mid a.b = b.a\}$

4. Consider the ring $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ under addition and multiplication of 2×2 matrices, then $A \in M_2(\mathbb{Z})$ is an unit if -

(a) $ad - bc \neq 0$ (c) and only if $ad - bc \neq 0$.

(b) $ad - bc$ is an even integer. (d) $ad - bc = \pm 1$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ exists only when $ad-bc \neq 0$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow$ multi identity $M_2(\mathbb{Z})$

5. Consider the following rings :
(i) $(\mathbb{Z}_5, +, \cdot)$ (ii) $(\mathbb{Z}_{15}, +, \cdot)$ (iii) $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication
(iv) $\mathbb{R}[x]$,Then

(a) (i), (iv) have no proper zero divisors.
(b) (i), (iii) have no proper zero divisors
(c) (i), (iii) have proper zero divisors.
(d) (i), (iii), (iv) have no proper zero divisors

$a \times_5 b = 0$ where $a \neq 0$, $b \neq 0$

$(\mathbb{Z}_5) = \{0, 1, 2, 3, 4\} \bmod 5$

No proper divisor of zero

$\mathbb{Z}_{15} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$

$3 \times 5 = 15 \equiv 0 \pmod{15}$

$R[x]$ also no proper divisor

$(1, 0) \times (0, 1) = (1 \times 0, 0 \times 1) = (0, 0)$

$\therefore$ $\mathbb{Z} \times \mathbb{Z}$ is a having proper divisor of 0

6. The number of units in the ring $\mathbb{Z}_{20}$ is
(a) 5 (b) 6 (c) 7 (d) 8

$\mathbb{Z}_{20} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{19}, \bar{11}, \bar{12}, \bar{13}, \bar{14}, \bar{15}, \bar{16}, \bar{17}, \bar{18}\}$

$3 \times 7 = 21 \equiv 1 \pmod{20} \Rightarrow \bar{3}$ & $\bar{7}$ are units

$1 \times 1 = 1 \pmod{20} \Rightarrow \bar{1}$ is also a unit

$9 \times 9 = 81 \equiv 1 \pmod{20} \Rightarrow 9$ is also a unit

$11 \times 11 = 121 \equiv 1 \pmod{20} \Rightarrow 11$ is also a unit

$13 \times 17 = 221 \equiv 1 \pmod{20} \Rightarrow 13$ & 17 are units

$19 \times 19 = 361 \equiv 1 \pmod{20} \Rightarrow 19$ is also unit

units $= \{\bar{1}, \bar{3}, \bar{7}, \bar{9}, \bar{11}, \bar{13}, \bar{17}, \bar{19}\}$

7. Which of the following is a subring of $(\mathbb{Q}, +, \cdot)$

(i) $R = \{a/b \,/\, a, b \in \mathbb{Z},\ (a, b) = 1, b \neq 0, b$ is not divisible by $3\}$.

(ii) $R = \{a/b \,/\, a, b \in \mathbb{Z},\ (a, b) = 1, b \neq 0, b$ is divisible by $3\}$.

(iii) $R = \{a/b \,/\, a, b \in \mathbb{Z},\ (a, b) = 1, b \neq 0, a$ is divisible by $3\}$.

(iii) $R = \{x^2 : x \in \mathbb{Q}\}$.

(a) (i) and (iv) (b) (ii) and (iv) ✓(c) (i) and (ii) (d) only (i).

If S is a subring of R, $a, b \in S \Rightarrow a-b \in S$
$\Rightarrow a.b \in S$

(i) $\frac{a_1}{b_1}$, $(a_1, b_1) = 1$, $b_1 \neq 0$, b_1 is not divisible by 3

$\frac{a_2}{b_2}$, $(a_2, b_2) = 1$, $b_2 \neq 0$, b_2 " " " 3

$\frac{a_1}{b_1}, \frac{a_2}{b_2} \in R$

$\frac{a_1}{b_1} - \frac{a_2}{b_2} = \frac{a_1 b_2 - b_1 a_2}{b_1 b_2}$, $b_1 b_2 \neq 0$, $b_1 b_2$ is also not divisible 3

$(a_1 b_2 - b_1 a_2, b_1 b_2) = 1$

$\frac{5}{7} - \frac{9}{11} = \frac{55-63}{77} = \frac{8}{77}$

$\frac{a_1 a_2}{b_1 b_2} \in R$, $b_1 b_1 \neq 0$, $(a_1 a_2, b_1 b_2) = 1$

(i), (ii) $\Rightarrow$ (c)

8. Let R and S be rings. Consider the ring $R \times S$ under component wise addition and multiplication.

(a) If R, S are integral domains, then $R \times S$ is an integral domain.
(b) $R \times S$ is an integral domain if and only if R, S are integral domains.
✓(c) $R \times S$ is not an integral domain, whatever R, S may be.
(d) $R \times S$ is not commutative even if R, S are commutative.

$(1, 0) \times (0, 1) = (0, 0) \Rightarrow$ is a divisor

Integral domain is not have any zero

$\Rightarrow R \times S$ is not an integral domain

9. Let R be an integral domain. Then, $x^2 = 1$

✓(a) has exactly two solutions. (b) may not have any solution.
(c) may have more than two solutions. (d) None of these.

$x^2 = 1 \Rightarrow x^2 - 1 = 0$

$\Rightarrow (x-1)(x+1) = 0$

[illegible] an integral domain $a.b = 0$

$\Rightarrow (x-1)(x+1) = 0$

But R is an integral domain $a \cdot b = 0$

$\Rightarrow a = 0$ & $b = 0$

$\therefore$ $x - 1 = 0 \Rightarrow x = 1$
$x + 1 = 0 \Rightarrow x = -1$

10. Consider the following rings: (i) $\mathbb{Z}_{18}$ (ii) $\mathbb{Z}_{12}$ (iii) $\mathbb{Z}_{10}$ (iv) $\mathbb{Z}_{14}$, then
(a) (i), (ii), (iii) ,(iv) have nilpotent elements. ✓ (b) (i), (ii) have nilpotent elements.
(c) (iii), (iv) have nilpotent elements. (d) None of these have nilpotent elements.

A element $a \in R$ is said to be nilpotent element of index k if $a^k = 0$

(i) $Z_{18} = \{ \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{10}, \bar{11}, \bar{12}, \bar{13}, \bar{14}, \bar{15}, \bar{16}, \bar{17} \}$ mod 18

Z_{18}: 18, 36, 54, 72,

$6^2 = 36 \equiv 0 \pmod{18} \Rightarrow 6$ is nilpotent ele of Z_{18}

$12^2 = 144 \equiv 0 \pmod{18} \Rightarrow 12$ is nilpotent

$Z_{12} \Rightarrow 6^2 = 36 \equiv 0 \pmod{12} \Rightarrow 6$ is nilpotent of Z_{12}

No nilpotent element Z_{10}, Z_{14}

Remaining Objectives

11. In an integral domain the number of elements which are their own inverses is
(a) 1 (b) 1 or 2 (c) 2 (d) infinitely many.

Solution :- x. x = 1 → $x^2 = 1$

an integral domain has at most two elements that satisfy the equation $x^2 = 1$.

As , $(x^2 - 1) = 0 \Rightarrow (x - 1)(x + 1) = 0$
As it is an integral domain then either
Either , x -1 = 0 or x + 1 = 0
Hence have two solutions .
Option (c)

12. In a ring $(\mathbb{Z}_n, +, \cdot)$ where n is a positive integer > 1

(i) $\bar{a}^2 = \bar{a} \Rightarrow \bar{a} = \bar{0}$ or $\bar{a} = \bar{1}$ for $\bar{a} \in \mathbb{Z}_n$.

(ii) $\bar{a} \cdot \bar{b} = \bar{0} \Rightarrow \bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ for $\bar{a}, \bar{b} \in \mathbb{Z}_n$.

(iii) $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$, $\bar{a} \neq 0 \Rightarrow \bar{b} = \bar{c}$ for $\bar{b}, \bar{c} \in \mathbb{Z}_n$. Then,

(a) the statements (i), (ii), (iii) are true.

(b) the statements (i) is true but (ii), (iii) may not be true.

(c) the statements (i), (ii), (iii) are true if n is prime.

(d) None of the above.

$(i)\ (a^2 - a) = 0 \Rightarrow\ a(a-1) = 0 \Rightarrow either\ a = 0$ or a = 1

(ii) a.b = 0 $\Rightarrow either\ a = 0$ or b= 0 when n is prime we get $\mathbb{Z}_n is$ $an\ integral\ domain$.

(iii) We know finite integral domain is a field $\mathbb{Z}_n is$ a field . All non zero elements has its inverse.

For any non zero a there exists a^{-1} such that a.b = a.c

$\Rightarrow a^{-1}.a.b = a^{-1}.a.c$

$\Rightarrow b = c$

Which is true for all prime n .

Option (c)

13. If R is a ring and a, b are zero divisors in R, then

(a) $a + b$ is always a zero divisor.

(b) $a + b$ is not a unit in R.

(c) $a + b$ may not be a zero divisor.

(d) None of these.

Solution :- Let $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are zero divisors of $M_2(\mathbb{R})$as $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ but $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is not a zero divisor of $M_2(\mathbb{R})$.

Option (c)

14. In the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}_2 \right\}$, the number of non-zero zero divisors is

(a) 6 (b) 7 (c) 5 (d) None of these.

Solution :-

R = { $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix},$ $\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}\}$

Here , $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} , \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

4 zero divisors .

Option (d)

15. If x is an idempotent element in $\mathbb{Z}_n$ $(x^2 = x)$, then
(a) $1 - x$ is a unit. (b) $1 + x$ is a unit.
(c) $1 - x$ is an idempotent. (d) None of these.

solution :-

As $x^2 = x$ in $\mathbb{Z}_n$ then

$x^2 - x = 0 \Rightarrow x(1 - x) = 0$

Now , $(1 - x)^2 = 1 - 2x + x^2 = 1 - 2x + x = 1 - x$

Therefore , (1-x) is an idempotent element . - Option (c)

17. Consider the rings $R_1 = (\mathbb{Z}_{10}, +, \cdot), R_2 = (\mathbb{Z}_{23}, +, \cdot), R_3 = M_2(\mathbb{Z})), R_4 = \mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
(a) R_1, R_2, R_3, R_4 are all integral domains. (b) Only R_2, R_3, R_4 are integral domains.
(c) R_2 is an integral domain. (d) R_2, R_4 are integral domains.

Solution :- (c)

As 23 is prime so $\mathbb{Z}_{23}$ is an integral domain.

But , 10 is not a prime so $\mathbb{Z}_{10}$is not an integral domain.

Also we know , $M_2(\mathbb{Z})$ and $\mathbb{Z} \times \mathbb{Z}$ has zero divisors so they are not inetgral domain.

18. Let R be an integral domain of characteristic p. Then,
(a) $(x + y)^m = x^m + y^m \ \forall x, y \in R$ if and only if $m = p$.
(b) $(x + y)^m = x^m + y^m \ \forall x, y \in R$ and $m = kp$.
(c) $(x + y)^{p^n} = x^{p^n} + y^{p^n} \ \forall x, y \in R$ and for all $n \in \mathbb{N}$.
(d) None of the above.

Solution :- (a)

$(x + y)^m = x^m + y^m \ \forall x , y \in R$ is possible only when m = p is a prime.

19. Consider the subset $S = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$ of $\mathbb{Z}_{10}$.
(a) S is a subring of $\mathbb{Z}_{10}$. (b) S is not a subring of $\mathbb{Z}_{10}$.
(c) S is a subring with multiplicative identity $\bar{6}$.
(d) S is a ring with multiplicative identity $\bar{6}$.

Solution :- (c)

S = { 0 , 2 , 4, 6 , 8 } mod 10

6 X 2 = 12 $\equiv 2 \ (mod\ 10)$

6 X 4 = 24 $\equiv 4\ (mod\ 10)$
6 X 6 = 36 $\equiv 6\ (mod\ 10)$
6 X 8 = 48 $\equiv 8\ (mod\ 10)$
Then S will be a subring with multiplication modulo 10 and identity is 6.
As a.b and a-b $\in S\ \forall\ a, b \in\ S$

20. Let R be a ring in which $x^2 = x$ for all $x \in R$. Then,
(a) R is an integral domain with characteristic 3.
(b) R is field with characteristic 3.
(c) Characteristic of R is 2. (d) None of these.

Solution :- (c)
B = { 0 , 1} for all x $\in R$, $x^2 = x$
Characteristic of a Boolean ring is always = 2.

21. The characteristics of the ring $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ under component wise addition and multiplication is
(a) 180 (b) 3 (c) 60 (d) 6

Solution :- (c)
Here , $\mathbb{Z}_{12} \times\ \mathbb{Z}_{15}$
Here characteristic is equal to .
l. c. m. = (12 , 15) = 60

22. $x \in \mathbb{R}[x]$ is
(a) is a unit in $\mathbb{R}[x]$.
(b) is a zero divisor in $\mathbb{R}[x]$.
(c) is neither a unit nor a zero divisor in $\mathbb{R}[x]$.
(d) None of these.

Solution :- (c)
As , $R\ [x]$ / (x) is isomprphic to $\mathbb{R}$ which a field then x is irreducible element i.e. a prime element in R[x] then x is not unit as well all zero divisor.

24. Which of the following is true
(a) $\mathbb{Z}_2[i], \mathbb{Z}_5[i]$ are integral domains and $\mathbb{Z}_3[i]$ is a field.
(b) $\mathbb{Z}_2[i], \mathbb{Z}_5[i]$ and $\mathbb{Z}_3[i]$ are fields
(c) $\mathbb{Z}_2[i], \mathbb{Z}_5[i]$ are fields and $\mathbb{Z}_3[i]$ is an integral domain.
(d) Only $\mathbb{Z}_2[i]$ is a field and $\mathbb{Z}_3[i], \mathbb{Z}_5[i]$ are integral domains.

Solution :- (b)
Here $\mathbb{Z}_2[\ i]$, $\mathbb{Z}_5[i]$ and $\mathbb{Z}_3[i]$
Here out of this three only all are finite integral domains .

As we know all the finite integral domain is a field . Hence $\mathbb{Z}_2[\,i], \mathbb{Z}_5[i]$ and $\mathbb{Z}_3[i]$ *all are field* .

25. If $H_{\mathbb{Z}} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z}\}$ then the multiplicative group of units of $H_{\mathbb{Z}}$ is

(a) $\{\pm 1\}$.

(b) $\{1, i, j, k\}$

(c) $\{\pm 1, \pm i, \pm j, \pm k\}$

(d) $H_{\mathbb{Z}} - \{0\}$.

Solution :- (c)

a + ib + cj + dk ∈ $H_{\mathbb{Z}}$

i×(-i) = 1 , (-1)×(-1) = 1 , j X(-j) = 1, k×(-k) = 1 , 1 ×1=1