Paper-I

Unit-III

Joint moment generating function

The concept of joint moment generating function (joint mgf) is a multivariate generalization of the concept of moment generating function. Similarly to the univariate case, a joint mgf uniquely determines the joint distribution of its associated random vector, and it can be used to derive the cross-moments of the distribution by partial differentiation.

The random variables X and Y defined on sample space S are (jointly) discrete if (X,Y) takes countable values.

Discrete variables X and Y are independent if the events {X=x} and {Y=y} are independent for all x and y.

The joint probability mass function $f_{X,Y}$: of X and Y is given by

$f(x, y) = f_{X,Y}(x,y) = P(X=x, Y=y)$.

The marginal pmf of X, $f_X(x)$, is found from:

$f(x) = f_X(x) = P(X=x)$

$= \sum_y P(X = x, Y = y)$

$= \sum_y f(x, y)$.

Similarly for $f_Y(y)$

X and Y are independent iff $f_{X,Y}(x, y) = f_X(x)\, f_Y(y)$, for all $x, y \in S$

The conditional probability mass function of Y given X=x, $f_{Y|X}(y|x)$, is defined by

$f_{Y|X}(y|x) = P(Y=y|\ X=x)$,for any x such that $P(x=x)>0$.

The random variables X and Yon S are called jointly continuous if their joint distribution function can be expressed as

$F_{X,Y}(x,y) = \int_{u=-\infty}^{x} \int_{v=-\infty}^{y} f(u, v)\, dv\, du,\ \ x, y \in S,$

Then $f_{X,Y}$ is the joint probability density function of X,Y.

If $f_{X,Y}$ is ‘sufficiently differentiable’ at (x,y) we have

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

The marginal pdf of X is $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

Similarly, the marginal pdf of Y is $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$.

X and Y are independent if

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y), \forall x, y \in S$$

Let X and Y be jointly distributed. The joint moment generating function of X and Y is $M(t_1, t_2) = E(e^{t_1 X + t_2 Y})$.

1. Let X and Y be jointly distributed continuous random variables with pdf f(x,y). The joint moment generating function of X and Y is

$$M(t_1, t_2) = E(e^{t_1 X + t_2 Y}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)\,dx\,dy$$

2. Let X and Y be jointly distributed discrete random variables with pdf f(x,y). The joint moment generating function of X and Y is

$$M(t_1, t_2) = E(e^{t_1 X + t_2 Y}) = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} e^{t_1 x + t_2 y} f(x,y)$$

Property 1. The mgfs of X and Y are $M(t_1, 0)$ and $M(0, t_2)$, respectively.

$$M(t_1, 0) = E(e^{t_1 X}) = M_X(t_1)$$

$$M(0, t_2) = E(e^{t_2 Y}) = M_Y(t_2)$$

Property 2. It determines the joint distribution of X and Y uniquely and it also yields the moments:

$$\frac{\partial^{m+n}}{(\partial t_1)^m (\partial t_2)^n} M(t_1, t_2) | t_1 = t_2 = 0 \quad = E(X^m Y^n).$$

$$\frac{\partial M(0,0)}{\partial t_1} = E(X) \qquad \frac{\partial M(0,0)}{\partial t_2} = E(Y)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) \qquad \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(Y^2)$$

$$\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} = E(XY)$$

Property 3. Suppose the joint mgf, $M(t_1, t_2)$, exists for the random variables X1 and X2. Then X1 and X2 are independent if and only if

$M(t_1, t_2) = M(t_1, 0)M(0, t_2)$;

that is, the joint mgf is identically equal to the product of the marginal mgfs.

Proof. If X_1 and X_2 are independent, then

$$\begin{aligned} M(t_1,t_2) &= E(e^{t_1X_1+t_2X_2}) \\ &= E(e^{t_1X_1}e^{t_2X_2}) \\ &= E(e^{t_1X_1})E(e^{t_2X_2}) \\ &= M(t_1,0)M(0,t_2). \end{aligned}$$

Thus the independence of X_1 and X_2 implies that the mgf of the joint distribution factors into the product of the moment-generating functions of the two marginal distributions.

Suppose next that the mgf of the joint distribution of X_1 and X_2 is given by $M(t_1,t_2) = M(t_1,0)M(0,t_2)$. Now X_1 has the unique mgf, which, in the continuous case, is given by

$$M(t_1,0) = \int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\,dx_1.$$

Similarly, the unique mgf of X_2, in the continuous case, is given by

$$M(0,t_2) = \int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\,dx_2.$$

Thus we have

$$\begin{aligned} M(t_1,0)M(0,t_2) &= \left[\int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\,dx_1\right]\left[\int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\,dx_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f_1(x_1)f_2(x_2)\,dx_1dx_2. \end{aligned}$$

We are given that $M(t_1,t_2) = M(t_1,0)M(0,t_2)$; so

$$M(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f_1(x_1)f_2(x_2)\,dx_1dx_2.$$

But $M(t_1,t_2)$ is the mgf of X_1 and X_2. Thus

$$M(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(x_1,x_2)\,dx_1dx_2.$$

The uniqueness of the mgf implies that the two distributions of probability that are described by $f_1(x_1)f_2(x_2)$ and $f(x_1,x_2)$ are the same. Thus

$$f(x_1,x_2) \equiv f_1(x_1)f_2(x_2).$$

That is, if $M(t_1,t_2) = M(t_1,0)M(0,t_2)$, then X_1 and X_2 are independent. This completes the proof when the random variables are of the continuous type. With random variables of the discrete type, the proof is made by using summation instead of integration. ■

By the uniqueness of the MGF (Property 1) we must have

$f\,(x_1, x_2) = f(x_1)\, f(x_2)$ for all (x_1, x_2)

It follows that X_1 and X_2 are independent. A similar proof is given in the case where (X_1, X_2) is of the discrete type.

The MGF technique uses the uniqueness property.

Property 4. $M\,(0, 0) = 1$ for any X and Y

Property 5. For c_1, c_2 and d_1, d_2 constants:

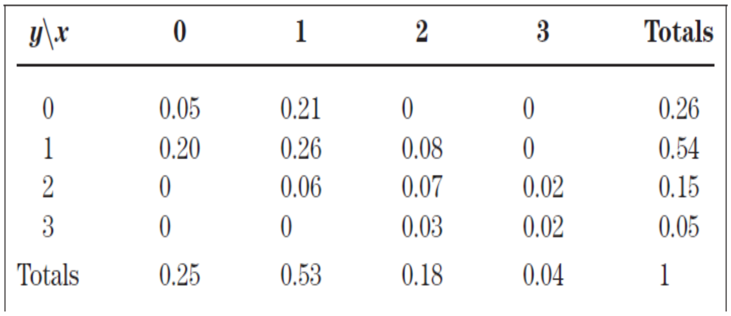
$M_{c1X+d1,\, c2Y+d2}\,(t_1, t_2) = e^{\,d1t1+d2t2}\, M_{X,\,Y}\,(c_1t_1, c_2t_2)$.

Proof. $M_{c1X+d1,\, c2Y+d2}\,(t_1, t_2) = E(e^{(c_1X+d_1)t_1+(c_2Y+d_2)t_2})$

$$= e^{d_1t_1+d_2t_2} E\,(e^{c_1Xt_1+c_2Yt_2})$$

$$= e^{d_1t_1+d_2t_2}\, M_{X,\,Y}\,(c_1t_1, c_2t_2)$$

Example 1. Each one of the r.v.'s *X* and *Y* takes on four values only, 0, 1, 2, 3, with joint Probabilities expressed best in a matrix form as in following Table.



$y \backslash x$	0	1	2	3	Totals
0	0.05	0.21	0	0	0.26
1	0.20	0.26	0.08	0	0.54
2	0	0.06	0.07	0.02	0.15
3	0	0	0.03	0.02	0.05
Totals	0.25	0.53	0.18	0.04	1

Solution: $M_{X,Y}\,(t1, t2) = M\,(t_1, t_2) = E(e^{t_1X+t_2Y}) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} e^{t_1x+t_2y} f(x, y)$

$= 0.05 + 0.20e^{t2} + 0.21e^{t1} + 0.26e^{\,t1+t2} + 0.06e^{\,t1+2t2} + 0.08\, e^{\,2t1+t2}$
$+ 0.07e^{\,2t1+2t2} + 0.03e^{\,2t1+3t2} + 0.02e^{\,3t1+2t2} + 0.02e^{\,3t1+3t2}$.

Example 2. Let the r.v.'s X and Y have the joint p.d.f.

$f(x, y) = \lambda_1\lambda_2\, e^{-\lambda 1x-\lambda 2y}$,

x, y > 0, $\lambda 1, \lambda 2 > 0$. For example, X and Y may represent the lifetimes of two components in an electronic system. Derive the joint MGF. Also find MGF of X and Y.

Solution: $M(t_1, t_2) = E(e^{t_1X+t_2Y}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)\,dx\,dy$

$$= \int_0^{\infty}\int_0^{\infty} e^{t_1x+t_2y}\ \lambda_1\lambda_2\, e^{-\lambda 1x-\lambda 2y}\, dx\, dy$$

$$= \int_0^{\infty} \lambda_1\, e^{-(\lambda_1-t_1)x}\, dx \int_0^{\infty} \lambda_2\, e^{-(\lambda_2-t_2)y}\, dy$$

$$= \left(-\frac{\lambda_1 e^{-(\lambda_1-t_1)x}}{\lambda_1-t_1}\right) \times \left(-\frac{\lambda_2 e^{-(\lambda_2-t_2)y}}{\lambda_2-t_2}\right) \qquad \text{[Range 0 to } \infty\text{]}$$

$$= \frac{\lambda_1}{\lambda_1-t_1} \times \frac{\lambda_2}{\lambda_2-t_2} \qquad \text{[provided } t_1 < \lambda_1 \text{ and } t_2 < \lambda_2\text{]}$$

The mgfs of X and Y are $M(t_1, 0)$ and $M(0, t_2)$, respectively.

$M(t_1, 0) = E(e^{t_1X}) = M_X(t_1) = \frac{\lambda_1}{\lambda_1-t_1}$

$M(t_2, 0) = E(e^{t_2Y}) = M_Y(t_2) = \frac{\lambda_2}{\lambda_2-t_2}$

Example 3. If $f(x1, x2) = e^{-x1-x2}$, $0 < x1 < \infty$, $0 < x2 < \infty$, zero elsewhere, is the joint pdf of the random variables X1 and X2, show that X1 and X2 are independent

Solution: $M(t_1, t_2) = E(e^{t_1X1+t_2X2})$

$$= \int_0^{\infty}\int_0^{\infty} e^{t_1X1+t_2X2}\, e^{-x1-x2}\, dx1dx2$$

$$= \int_0^{\infty} e^{-(1-t1)x1}\,\text{dx1} \int_0^{\infty} e^{-(1-t1)x1}\text{dx2}$$

$$= \frac{e^{-(1-t1)x1}}{-(1-t1)} \times \frac{e^{-(1-t2)x2}}{-(1-t2)} \qquad [0 < x1 < \infty, 0 < x2 < \infty]$$

$$= \frac{1}{1-t_1} \times \frac{1}{1-t_2} \qquad for\ \text{t2} < 1, \text{t1} < 1. \text{ [From Example 2]}$$

.

Accordingly, find the mean and the variance of Y = X1 + X2.

Example 4.

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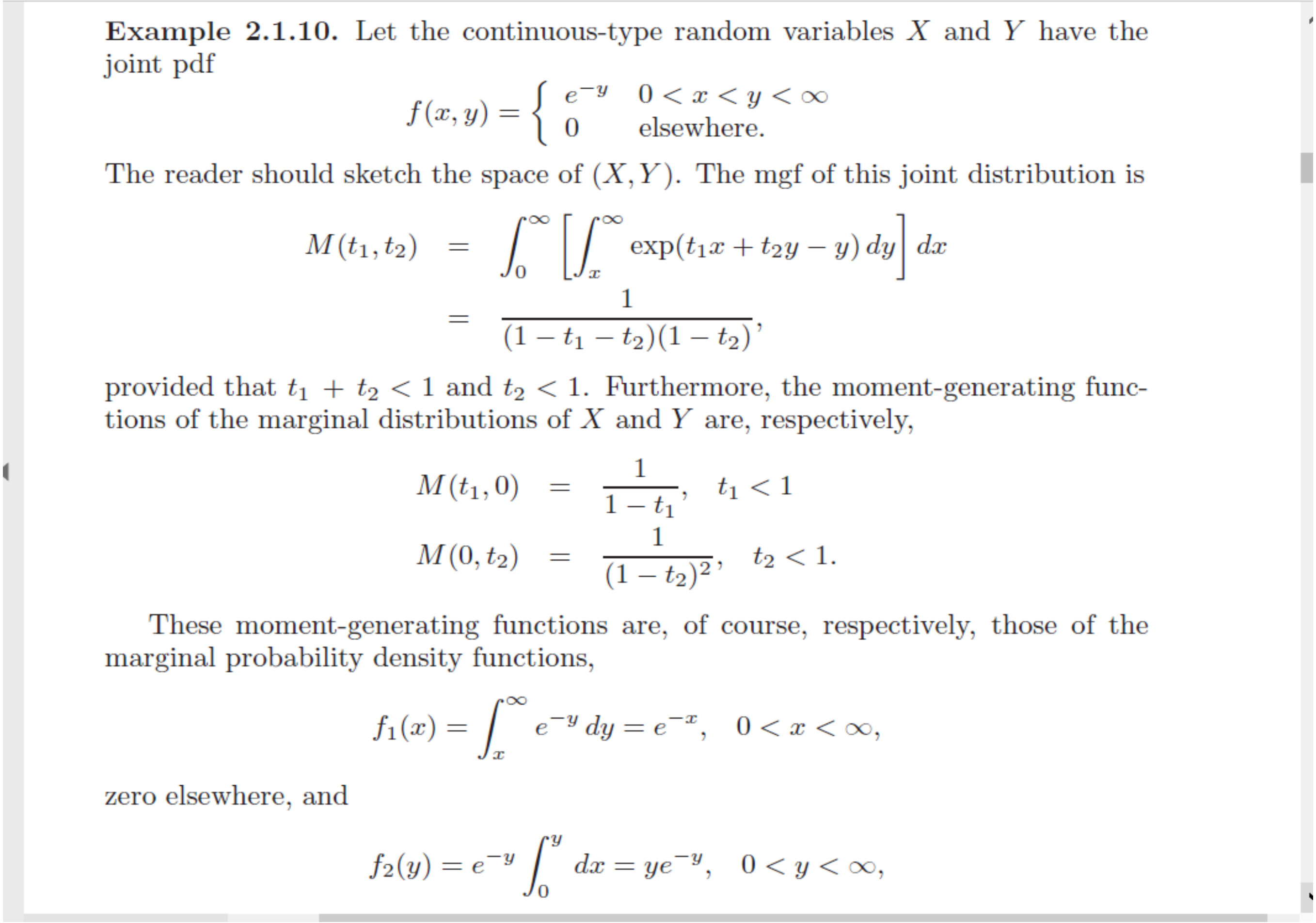


Example 2.1.10. Let the continuous-type random variables X and Y have the joint pdf

$$f(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

The reader should sketch the space of (X, Y). The mgf of this joint distribution is

$$\begin{aligned} M(t_1,t_2) &= \int_0^{\infty}\left[\int_x^{\infty} \exp(t_1x + t_2y - y)\,dy\right] dx \\ &= \frac{1}{(1-t_1-t_2)(1-t_2)}, \end{aligned}$$

provided that $t_1 + t_2 < 1$ and $t_2 < 1$. Furthermore, the moment-generating functions of the marginal distributions of X and Y are, respectively,

$$\begin{aligned} M(t_1,0) &= \frac{1}{1-t_1}, \quad t_1 < 1 \\ M(0,t_2) &= \frac{1}{(1-t_2)^2}, \quad t_2 < 1. \end{aligned}$$

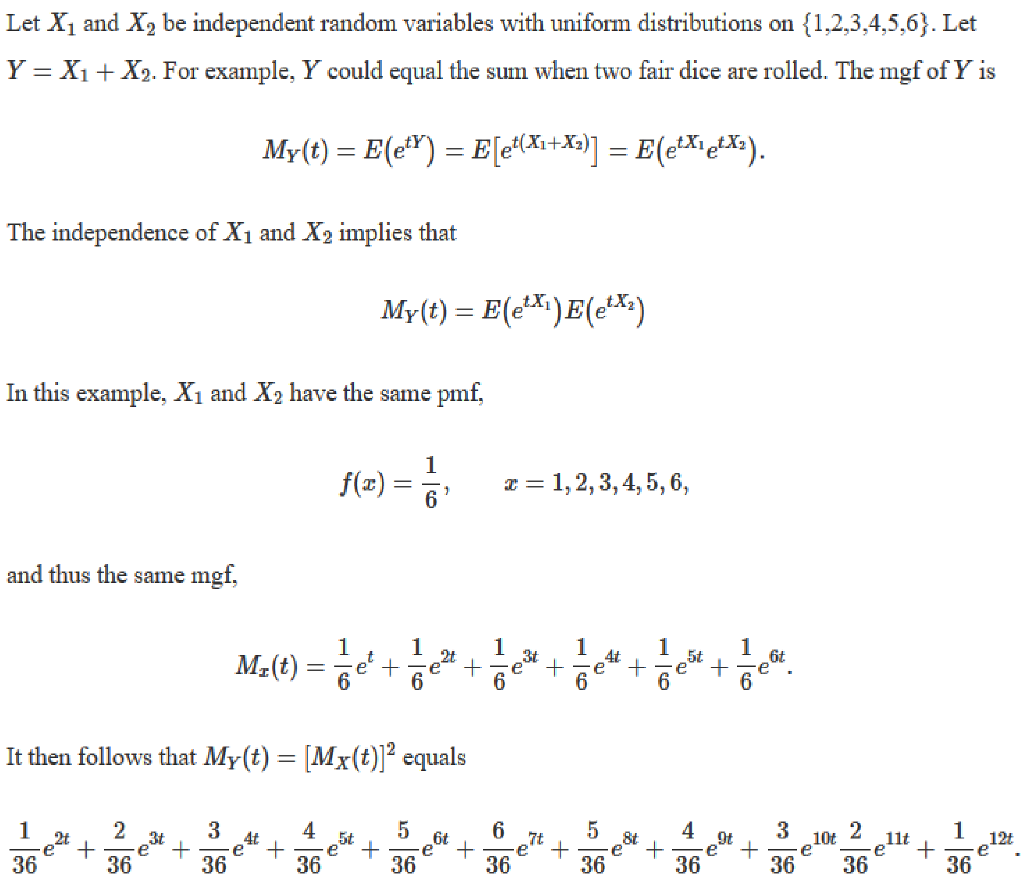
These moment-generating functions are, of course, respectively, those of the marginal probability density functions,

$$f_1(x) = \int_x^{\infty} e^{-y}\,dy = e^{-x}, \quad 0 < x < \infty,$$

zero elsewhere, and

$$f_2(y) = e^{-y}\int_0^y dx = ye^{-y}, \quad 0 < y < \infty,$$

Example 5.



Let X_1 and X_2 be independent random variables with uniform distributions on $\{1,2,3,4,5,6\}$. Let $Y = X_1 + X_2$. For example, Y could equal the sum when two fair dice are rolled. The mgf of Y is

$$M_Y(t) = E(e^{tY}) = E[e^{t(X_1+X_2)}] = E(e^{tX_1}e^{tX_2}).$$

The independence of X_1 and X_2 implies that

$$M_Y(t) = E(e^{tX_1})E(e^{tX_2})$$

In this example, X_1 and X_2 have the same pmf,

$$f(x) = \frac{1}{6}, \qquad x = 1,2,3,4,5,6,$$

and thus the same mgf,

$$M_x(t) = \frac{1}{6}e^t + \frac{1}{6}e^{2t} + \frac{1}{6}e^{3t} + \frac{1}{6}e^{4t} + \frac{1}{6}e^{5t} + \frac{1}{6}e^{6t}.$$

It then follows that $M_Y(t) = [M_X(t)]^2$ equals

$$\frac{1}{36}e^{2t} + \frac{2}{36}e^{3t} + \frac{3}{36}e^{4t} + \frac{4}{36}e^{5t} + \frac{5}{36}e^{6t} + \frac{6}{36}e^{7t} + \frac{5}{36}e^{8t} + \frac{4}{36}e^{9t} + \frac{3}{36}e^{10t}\frac{2}{36}e^{11t} + \frac{1}{36}e^{12t}.$$

Trinomial Distribution

Definition. Suppose we repeat an experiment n independent times, with each experiment ending in one of three mutually exclusive and exhaustive ways (success, first kind of failure, second kind of failure).

Now suppose that at each trial there are 3 possibilities, say “success”, “failure”, or “neither” of the two, with corresponding probabilities $p_1, p_2, 1 - p_1 - p_2$, which are the same for all trials.

Let X be the number of first type of trials and Y be the number of second type of trials. The joint distribution of (X,Y)is called the trinomial distribution.

$$f(x, y) = P(X{=}x, Y{=}y) = \frac{n!}{x!y!(n-x-y)!!}\, p_1^x\, p_2^y\, (1 - p_1 - p_2)^{n-x-y}$$

with: x = 0, 1, 2,……..n

y = 0, 1, 2,……..n

and $x + y \leq n$

p_1, p_2 are positive fractions such that $p_1 + p_2 < 1$.

Example:

1. Suppose n=20 students are selected at random:

- Let A be the event that a randomly selected student went to the football game on Saturday. Also, let P(A) = 0.20 = p_1, say
- Let B be the event that a randomly selected student watched the football game on TV on Saturday. Let P(A) = 0.50 = p_2, say
- Let C be the event that a randomly selected student completely ignored the football game on Saturday. Let P(C) = 0.30 = $1 - p_1 - p_2, say$

Joint Moment Generating Function:

$$M(t_1, t_2) = E(e^{t_1X+t_2Y}) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} e^{t_1x+t_2y} f(x,y)$$

$$= \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} e^{t_1x+t_2y} \frac{n!}{x!y!(n-x-y)!} p_1^x\, p_2^y\, (1-p_1-p_2)^{n-x-y}$$

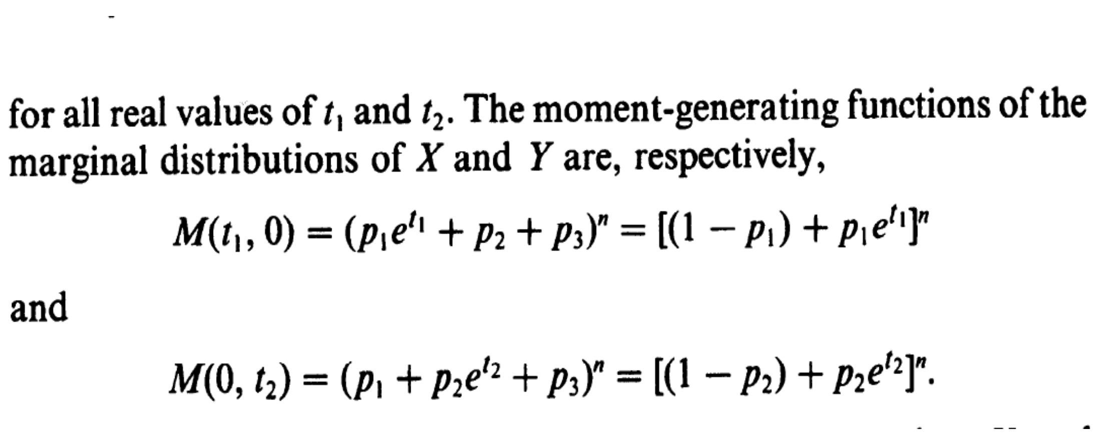
$$= \sum_{x=0}^{n}\sum_{y=0}^{n-x} \frac{n!}{x!y!(n-x-y)!} (p_1e^{t_1})^x (p_2\, e^{t_2})^y (1-p_1-p_2)^{n-x-y}$$

$$= \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} (p_1e^{t_1})^x \sum_{y=0}^{n-x} \frac{(n-x)!}{y!(n-x-y)!} (p_2\, e^{t_2})^y (1-p_1-p_2)^{n-x-y}$$

$$= \sum_{x=0}^{n} \frac{n!}{x!(n-x)!} (p_1e^{t_1})^x (p_2\, e^{t_2} + p_3)^{n-x}$$

[From Binomial expansion and by considering $1 - p_1 - p_2 = p_3$]

$$= (p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^n$$



for all real values of t_1 and t_2. The moment-generating functions of the marginal distributions of X and Y are, respectively,

$$M(t_1, 0) = (p_1e^{t_1} + p_2 + p_3)^n = [(1 - p_1) + p_1e^{t_1}]^n$$

and

$$M(0, t_2) = (p_1 + p_2e^{t_2} + p_3)^n = [(1 - p_2) + p_2e^{t_2}]^n.$$

X and Y are independent if and only if

$M(t_1, t_2) = M(t_1, 0)M(0, t_2)$;

that is, the joint mgf is identically equal to the product of the marginal mgfs.

Here $M(t_1, t_2) \neq M(t_1, 0)M(0, t_2)$

So, X and Y are not independent.

Marginal Distributions:

M $(t_1, t_2) = (p_1\ e^{t_1} + p_2\ e^{t_2} + p_3)^n$

So Moment Generating Function of X is M $(t_1, 0) = [(1 - p_1) + p_1e^{t_1}]^n$,

$\Rightarrow X{\sim}Bin(n, p_1)$

That is: f(x) = ${}^nC_x\ p_1^x(1 - p_1)^{n-x}$

with x = 0, 1, 2, …n

$\Rightarrow E(X) = Mean\ of\ X = \mu_1 = np_1$

and Var (X) = $\sigma_1^2 = np_1(1 - p_1)$

Moment Generating Function of Y is M $(0,\ t_2) = [(1 - p_2) + p_2e^{t_2}]^n$,

$\Rightarrow Y{\sim}Bin(n, p_2)$

That is: f(y) = ${}^nC_y\ p_2^y(1 - p_2)^{n-y}$

with y = 0, 1, 2, …n

$$\Rightarrow E(Y) = Mean\ of\ Y = \mu_2 = np_2$$

$$and\ Var\ (Y) = \ \sigma_2^2 = np_2(1 - p_2)$$

Alternative Method to find Moments:

$$\mu'_{mn} = \frac{\partial^{m+n}}{(\partial t_1)^m(\partial t_2)^n} \mathrm{M}\ (t_1, t_2)|t_1{=}t_2{=}0 \quad = \mathrm{E}\ (\mathrm{X}^m\mathrm{Y}^n).$$

$$\mu'_{10} = \frac{\partial M(0,0)}{\partial t_1} = E(X) \qquad \mu'_{01} = \frac{\partial M(0,0)}{\partial t_2} = E(Y)$$

$$\mu'_{20} = \frac{\partial^2 M(0,0)}{\partial t_1^2} = E(X^2) \qquad \mu'_{02} = \frac{\partial^2 M(0,0)}{\partial t_2^2} = E(Y^2)$$

$$\mu'_{11} = \frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} = E(XY)$$

Now $\mu'_{10} = \frac{\partial M(0,0)}{\partial t_1} = \frac{\partial M(t_1,t_2)}{\partial t_1}\ (t_1 = t_2 = 0)$

$$= \frac{\partial\ (p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}}}{\partial t_1}\ (t_1 = t_2 = 0)$$

$$= n(p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-1} p_1 e^{t_1} \qquad (t_1 = t_2 = 0)$$

$$= n(p_1 + p_2 + p_3)^{n-1} p_1$$

$$= np_1 \quad [Since\ p_1 + p_2 + p_3 = 1]$$

Similarly, Now $\mu'_{01} = \frac{\partial M(0,0)}{\partial t_2} = \frac{\partial M(t_1,t_2)}{\partial t_2}\ (t_1 = t_2 = 0)$

$$= \frac{\partial\ (p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}}}{\partial t_2}\ (t_1 = t_2 = 0)$$

$$= n(p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-1} p_2 e^{t_2} \qquad (t_1 = t_2 = 0)$$

$$= n(p_1 + p_2 + p_3)^{n-1} p_2$$

$$= np_2 \quad [Since\ p_1 + p_2 + p_3 = 1]$$

$$\mu'_{20} = \frac{\partial^2 M(0,0)}{\partial t_1^2} = \frac{\partial^2 M(t_1,t_2)}{\partial t_1^2} \quad (t_1 = t_2 = 0)$$

$$= \frac{\partial M'(t_1,t_2)}{\partial t_1}\ (t_1 = t_2 = 0)$$

$$= \frac{\partial [n(p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-1} p_1 e^{t_1}]}{\partial t_1}\ (t_1 = t_2 = 0)$$

$= np_1[e^{t_1}(n-1)(p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-2} p_1\, e^{t_1} + (p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-1} e^{t_1}]$

$= np_1[(n-1)(p_1 + p_2 + p_3)^{n-2} p_1 \ + (p_1 + p_2 + p_3)^{n-1}]$

$= np_1[(n-1)p_1 + 1] = np_1 + n(n-1)p_1^2$

$\mu_{20} = \mu'_{20} - (\mu'_{10})^2 \ = np_1 + n(n-1)p_1^2 \text{-} (np_1)^2$

$= np_1 - np_1^2 \ = np_1(1 - p_1)$

$Similarly, \mu'_{02} = \frac{\partial^2 M(0,0)}{\partial t_2^2} = \frac{\partial^2 M(t_1,t_2)}{\partial t_2^2} \quad (t_1 = t_2 = 0)$

$= \frac{\partial M'(t_1,t_2)}{\partial t_2} (t_1 = t_2 = 0)$

$= \frac{\partial[n(p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-1} p_2 e^{t_2}]}{\partial t_1} (t_1 = t_2 = 0)$

$= np_2[e^{t_2}(n-1)(p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-2} p_2\, e^{t_2} + (p_1\, e^{t_1} + p_2\, e^{t_2} + p_3)^{\mathrm{n}-1} e^{t_2}]$

$= np_2[(n-1)(p_1 + p_2 + p_3)^{n-2} p_2 \ + (p_1 + p_2 + p_3)^{n-1}]$

$= np_2[(n-1)p_2 + 1] = np_2 + n(n-1)p_2^2$

$\mu_{02} = \mu'_{02} - (\mu'_{01})^2 \ = np_2 + n(n-1)p_2^2 \text{-} (np_2)^2$

$= np_2 - np_2^2 \ = np_2(1 - p_2)$

Conditional Distributions:

If Y = y, then the conditional distribution of X/(Y = y) is Binomial $(\mathrm{n} - \mathrm{y}, \frac{\mathrm{p}_1}{1-\mathrm{p}_2})$

Proof: P(X=x/Y=y) = $\frac{\mathrm{P}((\mathrm{X=x}) \cap (\mathrm{Y=y}))}{\mathrm{P(Y=y)}}$

$$= \frac{\frac{n!}{x!y!(n-x-y)!!} p_1^x \, p_2^y \, (1-p_1-p_2)^{n-x-y}}{{}^{n}C_y \, p_2^y (1-p_2)^{n-y}}$$

$$= \frac{\frac{n!}{x!y!(n-x-y)!!} p_1^x \, p_2^y \, (1-p_1-p_2)^{n-x-y}}{\frac{n!}{y!(n-y)!} \, p_2^y (1-p_2)^{n-y}}$$

$$= \frac{(n-y)! p_1^x \ (1-p_1-p_2)^{n-x-y}}{x!(n-x-y)!(1-p_2)^{n-y}}$$

$$= \frac{(n-y)! p_1^x \ (1-p_1-p_2)^{n-x-y}}{x!(n-x-y)!(1-p_2)^{n-x-y+x}}$$

$$= {}^{n-y}C_x \left(\frac{p_1}{1-p_2}\right)^x \left(\frac{1-p_1-p_2}{1-p_2}\right)^{n-y-x}$$

$$= {}^{n-y}C_x \left(\frac{p_1}{1-p_2}\right)^x \left(1-\frac{p_1}{1-p_2}\right)^{n-y-x}$$

for x = 0;1; :::; (n-y). Hence the conditional distribution of X/(Y = y) is Binomal $(n-y, \frac{p_1}{1-p_2})$

⇒ E(X/(Y = y)) = $(n-y)\,\frac{p_1}{1-p_2}$

and V(X/(Y = y)) = $(n-y)\,\frac{p_1}{1-p_2}\left(1-\frac{p_1}{1-p_2}\right)$

E [XY] = E [YE[X/Y]] [From property of expectation]

$$= E\left[Y(n-Y)\,\frac{p_1}{1-p_2}\right]$$

$$= \frac{p_1}{1-p_2}\ E[Y(n-Y)]$$

$$= \frac{p_1}{1-p_2}\,[E\,(nY) - E(Y^2)]$$

$$= \frac{p_1}{1-p_2}\,[n\,E(Y) - E(Y^2)]$$

$$= \frac{p_1}{1-p_2}\ [n \times np_2 - \ np_2 - n(n-1)p_2^2]$$

[Since E$(Y^2) = \mu'_{02} = np_2 + n(n-1)p_2^2$]

$$= \frac{p_1}{1-p_2}\left[n^2p_2 - np_2 - n(n-1)p_2^2\right]$$

$$= \frac{p_1}{1-p_2}\left[np_2(n-1) - n(n-1)p_2^2\right]$$

$$= \frac{p_1}{1-p_2}\left[np_2(n-1)(1-p_2)\right]$$

$$= n(n-1)p_1p_2$$

Therefore Cov(X,Y) = E[XY]- E[X]E[Y]

$$= n(n-1)p_1p_2 - np_1 \times np_2$$

$$= (n^2-n)(p_1p_2) - n^2p_1p_2$$

$$= -np_1p_2$$

ρ(X, Y) = Correlation (X, Y) = $\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$

$$= \frac{-np_1p_2}{\sqrt{np_1(1-p_1)\times np_2(1-p_2)}}$$

$$= \frac{-np_1p_2}{n\sqrt{p_1p_2}\sqrt{(1-p_1)(1-p_2)}} \qquad = - \frac{\sqrt{p_1p_2}}{\sqrt{(1-p_1)(1-p_2)}}$$

$$= -\sqrt{\frac{p_1p_2}{(1-p_1)(1-p_2)}}$$

Note that if $p_1 + p_2 = 1$ then Y = n-X and there is an exact linear relation between Y and X. In this case it is easily seen that ρ(X,Y) = -1.

From Property 5 of Joint Moment Generating Function, we get,

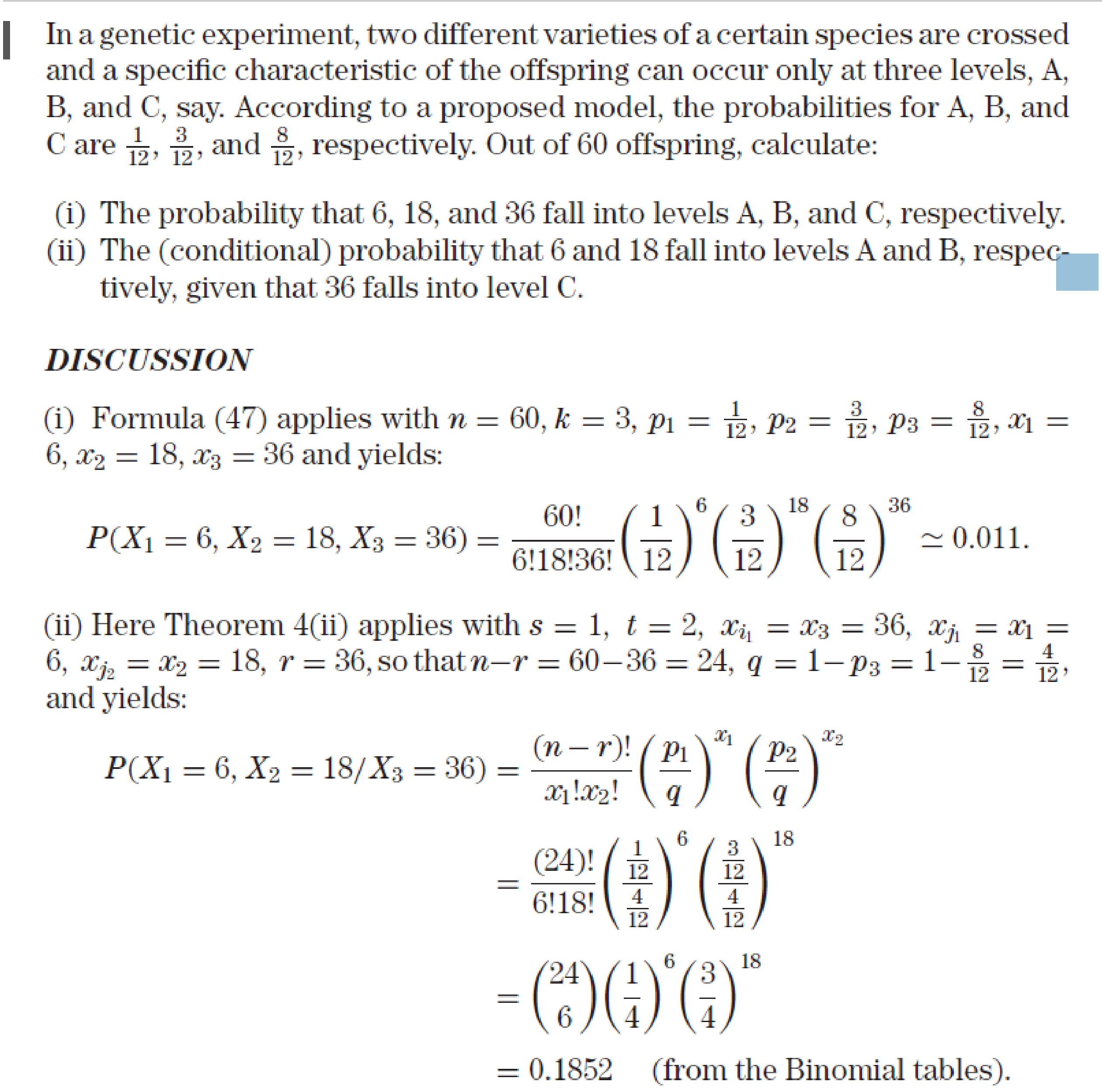
For c_1, c_2 and d_1, d_2 constants:

$M_{c1X+d1,\, c2Y+d2}(t_1, t_2) = e^{d1t1+d2t2} M_{X,\,Y}(c_1t_1, c_2t_2)$.

$\Rightarrow M_{X,\,Y2}(t_1, t_2) = M_{X,\,Y}(c_1t_1, c_2t_2)$.

if X and Y are independent and X has MGF $M_X(t)$ and Y has MGF $M_Y(t)$, then the MGF of $X+Y$ is just $M_X(t)M_Y(t)$, or the product of the two MGFs.

Example 1:



In a genetic experiment, two different varieties of a certain species are crossed and a specific characteristic of the offspring can occur only at three levels, A, B, and C, say. According to a proposed model, the probabilities for A, B, and C are $\frac{1}{12}$, $\frac{3}{12}$, and $\frac{8}{12}$, respectively. Out of 60 offspring, calculate:

(i) The probability that 6, 18, and 36 fall into levels A, B, and C, respectively.

(ii) The (conditional) probability that 6 and 18 fall into levels A and B, respectively, given that 36 falls into level C.

DISCUSSION

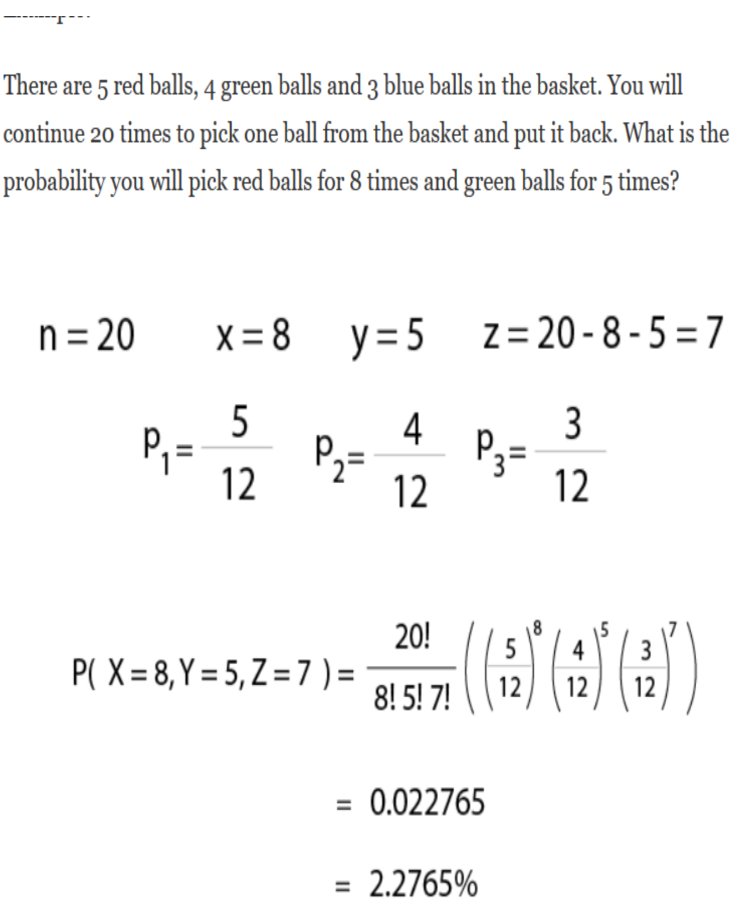
(i) Formula (47) applies with $n = 60$, $k = 3$, $p_1 = \frac{1}{12}$, $p_2 = \frac{3}{12}$, $p_3 = \frac{8}{12}$, $x_1 = 6$, $x_2 = 18$, $x_3 = 36$ and yields:

$$P(X_1 = 6, X_2 = 18, X_3 = 36) = \frac{60!}{6!18!36!}\left(\frac{1}{12}\right)^{6}\left(\frac{3}{12}\right)^{18}\left(\frac{8}{12}\right)^{36} \simeq 0.011.$$

(ii) Here Theorem 4(ii) applies with $s = 1$, $t = 2$, $x_{i_1} = x_3 = 36$, $x_{j_1} = x_1 = 6$, $x_{j_2} = x_2 = 18$, $r = 36$, so that $n - r = 60 - 36 = 24$, $q = 1 - p_3 = 1 - \frac{8}{12} = \frac{4}{12}$, and yields:

$$\begin{aligned} P(X_1 = 6, X_2 = 18/X_3 = 36) &= \frac{(n-r)!}{x_1!x_2!}\left(\frac{p_1}{q}\right)^{x_1}\left(\frac{p_2}{q}\right)^{x_2} \\ &= \frac{(24)!}{6!18!}\left(\frac{\frac{1}{12}}{\frac{4}{12}}\right)^{6}\left(\frac{\frac{3}{12}}{\frac{4}{12}}\right)^{18} \\ &= \binom{24}{6}\left(\frac{1}{4}\right)^{6}\left(\frac{3}{4}\right)^{18} \\ &= 0.1852 \qquad \text{(from the Binomial tables).} \end{aligned}$$

Example 2:



There are 5 red balls, 4 green balls and 3 blue balls in the basket. You will continue 20 times to pick one ball from the basket and put it back. What is the probability you will pick red balls for 8 times and green balls for 5 times?

$$n = 20 \qquad x = 8 \qquad y = 5 \qquad z = 20 - 8 - 5 = 7$$

$$p_1 = \frac{5}{12} \qquad p_2 = \frac{4}{12} \qquad p_3 = \frac{3}{12}$$

$$P(X = 8, Y = 5, Z = 7) = \frac{20!}{8!\,5!\,7!}\left(\left(\frac{5}{12}\right)^8\left(\frac{4}{12}\right)^5\left(\frac{3}{12}\right)^7\right)$$

$$= 0.022765$$

$$= 2.2765\%$$

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https://www.youtube.com/watch?v=syVW7DgvUaY

Multinomial Distribution

Definition

Multinomial distribution is the generalization of binomial distribution. Consider k events E1, E2,…, Ek. The event E1 occurs X1 times, E2 occurs X2 times and so on, with the corresponding probability p1, p2, …pk respectively.

i = 1, 2, 3, …,k. $p_1 + p_2 + \cdots .. + p_{k-1} + p_k = 1$

Then the joint probability function of k events is given by,

$$f\,(x_1, x_2, \ldots .. x_k)) = P(X_1 = x_1, X_2 = x_2, \ldots .. X_k = x_k)$$

$$= \frac{n!}{x_1!\ x_2! \ldots\ldots x_k!}\, p_1^{x_1}\, p_2^{x_2} \ldots \ldots \ldots \ldots .\, p_k^{x_k} = \frac{n!}{\prod_{i=1}^{k} x_i!} \prod_{i=1}^{k} p_i^{x_i}$$

with: x_i = 0, 1, 2,……..n so that $x_1 + x_2 + \cdots \dots\dots x_k = n$

Alternatively,

Let $x_1, x_2 \dots\dots..x_{k-1}$ be nonnegative integers such that

$$x_1 + x_2 + \cdots \dots\dots x_{k-1} \leq n$$

Then the probability that exactly x_i trials terminate in A_i, i = 1, 2, ... , k- l, and hence that Xk = n - (x1 + x2 + · · · + Xk-1) trials terminate in Ak is clearly

$$= \frac{n!}{x_1!\ x_2!\dots\dots x_k!} p_1^{x_1}\ p_2^{x_2} \dots\dots\dots\dots.p_k^{x_k}$$

Definition. An RV $(X_1, X_2, \dots\dots..X_{k-1})$ with joint PMF given by
P{X1 = x1, X2 = x2, .. . , Xk-1 = xk-1)

f $(x_1, x_2, \dots\dots\dots x_{k-1})$

$$= \frac{n!}{x_1!\ x_2!\dots\dots x_{k-1}!(n-x_1-x_2- \quad -x_{k-1})!} p_1^{x_1}\ p_2^{x_2} \dots\dots\dots\dots.p_k^{n-x_1-x_2- \quad -x_{k-1}}$$

$$== \frac{n!}{x_1!\ x_2!\dots\dots x_{k-1}!(n-x_1-x_2- \quad -x_{k-1})!}$$

$$p_1^{x_1}\ p_2^{x_2} \dots..(1 - p_1 - \cdots \dots - p_{k-1})^{(n-x_1-\cdots\dots..x_{k-1})}$$

$$\text{If } x_1 + x_2 + \cdots \dots\dots x_{k-1} \leq n$$

For example, if a fair die is tossed twelve times, the probability of getting 1, 2, 3, 4, 5 and 6 points exactly twice each is given by

P $(x_1 = 2, x_2 = 2, x_3 = 2, x_4 = 2, x_5 = 2, x_6 = 2)$

$$= \frac{12!}{2!2!2!2!2!2!} \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2 \left(\frac{1}{6}\right)^2$$

=0.00344

MGF of Multinomial Distribution

M $(t_1, t_2, \dots t_{k-1})$ =E $(e^{t_1X_1 + t_2X_2 + \cdots\dots.t_{k-1}X_{k-1}})$

$$= \sum_{x_1+x_2+\cdots\dots\dots x_{k-1}=0}^{n} e^{t_1X_1 + t_2X_2 + \cdots\dots.t_{k-1}X_{k-1}} \frac{n!}{x_1!x_2!\dots\dots\dots x_{k!}} p_1^{x_1}\ p_2^{x_2} \dots\dots\dots p_k^{x_k}$$

=

$$\sum_{x_1+x_2+\cdots\ldots\ldots x_{k-1}=0}^{n} \frac{n!}{x_1!\,x_2!\ldots\ldots\ldots x_{k!}} (p_1 e^{t_1})^{x_1} (p_2 e^{t_2})^{x_2} \ldots\ldots (p_{k-1} e^{t_{k-1}}) p_k^{x_k}$$

$$= (p_1\, e^{t_1} + p_2\, e^{t_2} + \cdots \ldots\ldots\ldots p_{k-1} e^{t_{k-1}} + p_k)^{\text{n}}$$

for all $t_1, t_2, \ldots\ldots . t_{k-1} \in$ R.

$$M(t_1, 0, 0, \ldots, 0) = (p_1 e^{t_1} + p_2 + \cdots + p_k)^n = (1 - p_1 + p_1 e^{t_1})^n,$$

which is binomial. Indeed, the marginal PMF of each X_i, $i = 1, 2, \ldots, k-1$, is binomial. Similarly, the joint MGF of X_i, X_j, $i, j = 1, 2, \ldots, k-1$ $(i \neq j)$, is

$$M(0, 0, \ldots, 0, t_i, 0, \ldots, 0, t_j, 0, \ldots, 0) = [p_i e^{t_i} + p_j e^{t_j} + (1 - p_i - p_j)]^n,$$

which is the MGF of a *trinomial distribution* with PMF

$$(58)\quad f(x_i, x_j) = \frac{n!}{x_i!\,x_j!\,(n - x_i - x_j)!} p_i^{x_i} p_j^{x_j} p_k^{n - x_i - x_j}, \qquad p_k = 1 - p_i - p_j.$$

Note that the RVs $X_1, X_2, \ldots, X_{k-1}$ are dependent.

From the MGF of $(X_1, X_2, \ldots, X_{k-1})$ or directly from the marginal PMFs we can compute the moments. Thus

$$(59)\quad EX_j = np_j \quad \text{and} \quad \text{var}(X_j) = np_j(1 - p_j), \qquad j = 1, 2, \ldots, k-1,$$

and for $j = 1, 2, \ldots, k-1$, and $i \neq j$,

$$(60)\qquad \text{cov}(X_i, X_j) = E[(X_i - np_i)(X_j - np_j)] = -np_i p_j.$$

It follows that the correlation coefficient between X_i and X_j is given by

$$(61)\quad \rho_{ij} = -\left[\frac{p_i p_j}{(1 - p_i)(1 - p_j)}\right]^{1/2}, \qquad i, j = 1, 2, \ldots, k-1 \quad (i \neq j).$$

We can consider the marginal of $X_{j+1}, X_{j+2}, \ldots\ldots\ldots . X_{k-1}$

$$M\,(t_{j+1}, t_{j+2}, \dots t_{k-1})$$
$$= (p_1 \;+ p_2 \;+ \cdots \dots \dots + p_j + p_{j+1}e^{t_{j+1}} + \cdots \dots + p_{k-1}e^{t_{k-1}} + p_k)^{n}$$

$$= (p_{j+1}e^{t_{j+1}} + \cdots \dots + p_{k-1}e^{t_{k-1}} + (1 - \; p_{j+1} - p_{j+2} - \cdots .. - p_{k-1}\;))^{n}$$

$$f\,(x_{j+1}, x_{j+2}, \dots \dots \dots . x_{k-1})$$

$$= \frac{n!}{x_{j+1}!\; x_{j+2}! \dots\dots x_{k-1}! \left(n - x_{j+1} - x_{j+2} - \quad - x_{k-1}\right)!}$$

$$p_{j+1}^{x_{j+1}}\; p_{j+2}^{x_{j+2}} \dots \dots \dots \dots p_{k-1}^{x_{k-1}} \left(1 - p_{j+1} \;\; - p_{j+2} - \cdots . - p_{k-1}\right)^{n - x_{j+1} - \cdots x_{k-1}}$$

Conditional Distributions

$$f\,(x_1, x_2, \dots .. x_j / x_{j+1}, x_{j+2}, \dots \dots \dots . x_{k-1})$$

$$= \frac{f(x_1, x_2, \dots\dots\dots x_{k-1})}{f(x_{j+1}, x_{j+2}, \dots\dots\dots x_{k-1})}$$

$$= \frac{\dfrac{n!}{x_1!\; x_2! \dots\dots x_{k-1}! (n - x_1 - x_2 - \quad - x_{k-1})!}}{\dfrac{n!}{x_{j+1}!\; x_{j+2}! \dots\dots x_{k-1}! \left(n - x_{j+1} - x_{j+2} - \quad - x_{k-1}\right)!}}$$

$$\frac{p_1^{x_1}\; p_2^{x_2} \dots .. (1 - p_1 - \cdots \dots - p_{k-1})^{(n - x_1 - \cdots \dots x_{k-1}\;)}}{p_{j+1}^{x_{j+1}}\; p_{j+2}^{x_{j+2}} \dots \dots \dots \dots p_{k-1}^{x_{k-1}} \left(1 - p_{j+1} \;\; - p_{j+2} - \cdots . - p_{k-1}\right)^{n - x_{j+1} - \cdots x_{k-1}}}$$

$$= \frac{(n - x_{j+1} - x_{j+2} - \quad - x_{k-1})!}{x_1!\; x_2! \dots\dots x_j!\; (n - x_1 - x_2 - \quad - x_{k-1})!} \; -$$