Practical no 1. Normal subgroups and Quotient groups

1. Let $H_1 = \{I, (12)\}$ and $H_2 = \{I, (123), (132)\}$. Then
   (a) H_1, H_2 are normal subgroups of S_3.
   (b) H_1 is a normal subgroup of S_3 but H_2 is not a normal subgroup of S_3.
   (c) H_1, H_2 are not normal subgroups of S_3
   (d) H_2 is a normal subgroup of S_3 but H_1 is not a normal subgroup of S_3.

2. Let $H_1 = \{\sigma \in S_n : \sigma(n) = n\}, H_2 = \{\sigma \in S_n : \sigma(k) = k, \text{ for some } k,\ 1 \leq k \leq n\}$. Then
   (a) H_1, H_2 are normal subgroups of S_n.
   (b) H_1 is a normal subgroup of S_n but H_2 is not a normal subgroup of S_n.
   (c) H_1, H_2 are not normal subgroups of S_n
   (d) H_2 is a normal subgroup of S_n but H_1 is not a normal subgroup of S_n.

3. Let $G = \dfrac{\mathbb{Z}}{20\mathbb{Z}}, H = \dfrac{4\mathbb{Z}}{20\mathbb{Z}}$ (under addition). Then order of quotient group $\dfrac{G}{H}$ is

   (a) 4 (b) ∞ (c) 5 (d) 20

4. Let H be a normal subgroup of G. Let $|aH| = 3$ in $\frac{G}{H}$ and $\circ(H) = 10$, then order of a is
   (a) 1 (b) 30
   (c) one of 3, 6, 15 or 30 (d) none of these.

5. Let G be a group of order 5. If $\Phi : \mathbb{Z}_{30} \to G$ is a group homomorphism, then ker Φ has order
   (a) 5 (b) 30 or 6 (c) 30 or 5 (d) 1

6. Let G be a finite group. If $f_1 : G \to \mathbb{Z}_{10}$ and $f_2 : G \to \mathbb{Z}_{15}$ are onto group homomorphisms, then order of G is
   (a) $30k$, where $k \in \mathbb{N}$ (b) 5^k, where $k \in \mathbb{N}$ (c) 10 or 15 (d) 5

7. In the quotient group $\dfrac{\mathbb{Z}_{18}}{<\bar{6}>}$ (under addition), the order of the element $\bar{5}+ <\bar{6}>$ is

   (a) 5 (b) 6 (c) 2 (d) 3

8. Let H be a subgroup of order 29 of a group G. If K is a subgroup of H, then
   (a) K is abelian and normal subgroup of G.
   (b) K is normal subgroup of H.
   (c) K is cyclic but may not be a normal subgroup of H .
   (d) H is normal subgroup of G and K is normal subgroup G.

9. Let $G = GL_2(\mathbb{R}), K = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\}, H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$. Then
   (a) H is a normal subgroup of K and K is a normal subgroup of G.
   (b) H is a normal subgroup of K but K is not a normal subgroup of G.

(c) H is a not normal subgroup of K but K is a normal subgroup of G.

(d) None of these.

10. Let H be a normal subgroup of a finite group G. If $|H| = 2$ and G has an element of order 3 then

(a) G has a cyclic subgroup of order 6.

(b) G has a non-abelian subgroup of order 6.

(c) G has subgroup of order 4.

(d) None of these.

11. Let G be a group of order 30. If $Z(G)$ has order 5, then
(a) $\frac{G}{Z(G)}$ is cyclic. (b) $\frac{G}{Z(G)}$ is abelian but not cyclic.
(c) $\frac{G}{Z(G)}$ is non-abelian. (d) None of these.

12. Let $G = GL_2(\mathbb{R})$, $H = \{A \in G : detA \in \mathbb{Q}\}$, then
(a) H is a normal subgroup of G. (b) H is not a subgroup of G.
(c) H is a subgroup which is not normal in G. (d) $H \subseteq Z(G)$.

13. Let $G = GL_2(\mathbb{R})$, $H = \{A \in G : detA = 2^m 3^n, \text{ for some } m, n \in \mathbb{Z}\}$, then
(a) H is a normal subgroup of G. (b) H is not a subgroup of G.
(c) H is a subgroup which is not normal in G. (d) $H \subseteq Z(G)$.

14. Let $G = U(16)$, $H = \{\bar{1}, \bar{15}\}$, $K = \{\bar{1}, \bar{9}\}$, then

(a) H, K are isomorphic groups and $\frac{G}{H}, \frac{G}{K}$ are isomorphic groups.

(b) H, K are not isomorphic groups but $\frac{G}{H}, \frac{G}{K}$ are isomorphic groups.

(c) H is not isomorphic to K.

(d) $\frac{G}{H}, \frac{G}{K}$ are not isomorphic groups.

15. Let $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in 2\mathbb{Z} \right\}$, $G = M_2(\mathbb{Z})$, under addition of 2×2 matrices. The quotient group $\frac{G}{H}$ has
(a) 4 elements (b) 16 elements (c) 12 elements (d) 8 elements

16. Let $G = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, a^4 = e = b^2, aba = b, H = \{e, b, a^2b, a^2\}, K = \{e, b\}$
(a) K is normal in H and H is normal in G. (b) K is not normal in H.
(c) K is normal in G. (d) H is not normal in G.

17. The quotient group $\left(\frac{\mathbb{Q}}{\mathbb{Z}}, +\right)$ is
(a) an infinite group in which only identity is of finite order.
(b) is an infinite cyclic group of finite index.

(c) an infinite group in which every element is of finite order.
(d) None of these.

18. Let G be a non-Abelian group of order pq, where p and q are distinct primes then
(a) $\circ(Z(G)) = p$ (b) $\circ(Z(G)) = q$
(c) $Z(G) = \{e\}$ (d) None of these.

19. If H is any non-trivial sugroup of a cyclic G then G/H
    (a) is infinite if G is infinite.
    (b) is finite
    (c) is not cyclic
    (d) None of these.

20. If G be an Abelian group then $H = \{(g, g) : g \in G\}$ is
    (a) normal in $G \times G$.
    (b) is not normal in $G \times G$.
    (c) is not a subgroup of $G \times G$
    (d) None of these.

21. The index of centre of a finite non-Abelian group
    (a) is $\circ(G)$.
    (b) is a prime
    (c) can not be a prime
    (d) None of these.

22. If N is a normal subgroup of G and all the elements of G/N and N have finite order, then
    (a) every element of G has finite order.
    (b) every element of G has infinite order.
    (c) G can have elements of infinite order.
    (d) None of these.

23. If H is a subgroup of S_n having order $n!/2$, then which of the following is not true
    (a) H is normal in S_n
    (b) $\sigma^2 \in H$ for every $\sigma \in S_n$.
    (c) H contains all 3-cycles.
    (d) $H \neq A_n$.

Practical 1 Descriptive Question

1. Let $G = GL_2(\mathbb{R}), K = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Prove or disprove: K is normal subgroup of G.

2. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\}, H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$. Prove that (i) H is a normal subgroup of G. (ii) $\frac{G}{H}$ is abelian.

3. Find the order of $\bar{5} + \langle \bar{14} \rangle$ in $\frac{\mathbb{Z}_{42}}{\langle \bar{14} \rangle}$.

4. Find the order of $\bar{14} + \langle \bar{8} \rangle$ in $\frac{\mathbb{Z}_{24}}{\langle \bar{8} \rangle}$.

5. In the following examples show that K is a normal subgroup of H and H is a normal subgroup of G, but K is not a normal subgroup of G.

   (i) $G = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, a^4 = e = b^2, aba = b, H = \{e, b, a^2b, a^2\}$, $K = \{e, b\}$.

   (ii) $G = A_4, H = \{I, (12)(34), (13)(24), (14)(23)\}, K = \{I, (12)(34)\}$.

6. Let $\mathbb{Q}_8 = \{\pm 1, \pm i, \pm j, \pm k\}, i^2 = j^2 = k^2 = -1 = ijk$. Show that

   (i) $\mathbb{Z}(\mathbb{Q}_8) = \{1, -1\}$.

   (ii) Every subgroup of $\mathbb{Q}_8$ is normal in $\mathbb{Q}_8$.

7. Let $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in 2\mathbb{Z} \right\}, G = M_2(\mathbb{Z})$, under addition of 2×2 matrices. Find order of the quotient group $\frac{G}{H}$ and describe $\frac{G}{H}$.

8. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0 \right\}, H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$. Prove that H is a normal subgroup of G and $\frac{G}{H} \cong (\mathbb{R}^+, \dot{})$. (the group of positive real numbers under multiplication).

9. Show that $\frac{\mathbb{R}^*}{\{1, -1\}} \cong \mathbb{R}^+$, for the multiplicative groups $\mathbb{R}^* = \mathbb{R} - \{0\}$, $\mathbb{R}^+$ of positive reals.

10. Show that A_4 has no subgroup of order 6.

11. Show that order of each element of the quotient group $\frac{\mathbb{Q}}{\mathbb{Z}}$ is finite.

12. Let G be a cyclic group of order 36 generated by a. Let $H = \langle a^6 \rangle$. Describe the quotient group $\frac{G}{H}$.

13. $G = A_4, K = \{I, (12)(34), (13)(24), (14)(23)\}$. Show that $\frac{A_4}{K} \cong A_3$.

14. Let H be a normal subgroup of S_4, $\circ(H) = 4$. Prove that $\frac{S_4}{H} \cong S_3$.

15. Show that $\frac{\mathbb{Q}}{\mathbb{Z}}$ has a unique subgroup of order n for each positive integer n.

16. Let G be a finite abelian group of order n. If $x^3 = e \; \forall x \in G$, show inductively that the order of G is 3^k for some $k \in \mathbb{N} \cup \{0\}$.

17. Let K be a cyclic subgroup of a group G which is normal in G. Show that any subgroup H of K is a normal subgroup of G.

18. Let G be a subgroup s.t. $(ab)^n = a^n b^n$ for some position integer n. Show that $G(n) = \{x^n \,/\, x \in G\}$ is a normal subgroup of G.

19. Let H and K be subgroup of a group G such that $H \cap K = \{e\}$ then show that $hk = kh$, $h \in H \; , \; k \in K$.

20. Suppose $G/Z(G)$ is cyclic then prove that G is Abelian. Further if G is a group of order 30 and $Z(G)$ has order 5 Show that $G/Z(G)$ is cyclic.

21. Let H be a normal subgroup of G of order 2. Show that $H \subseteq Z(G)$. Further if G is of order 10 show that G is Abelian.

22. If H is a subgroup of G such that $x^2 \in H$ for each $x \in G$ then show that H is a subgroup of G and G/H is Abelian.

23. Prove that the map $\theta : GL_2(\mathbb{R}) \to (\mathbb{R}^*, \cdot)$ given by $\theta(A) = det\, A$ is an onto homomorphism. Prove $SL_2(\mathbb{R})$ is a normal subgroup of $GL_2(\mathbb{R})$.

24. Let G be a subgroup and $H = \{g^2 \,/\, g \in G\}$ is a subgroup of G. Show H is normal in G.

25. Let H be a normal subgroup of a finite group G. If G/H has an elements of order n show that G has an element of order n.

26. Let $G =< a >$ be a cyclic group of order 21. Let$H =< a^7 >$. Find the order of element a^5H in the quotient group G/H.

27. Let $G =< a >$ be a cyclic group of order 24. Let $H =< a^{12} >$ and $K =< a^6 >$.

    (i) In G/H, find orders of a^2H, a^3H, a^4H, a^5H.

    (ii) In G/K, find orders of a^2K, a^3K, a^4K, a^5K.

28. Show that the map $\phi : \mathbb{Q} \to S_1$ defined by $\phi(m/n) = e^{2\pi mi/n}$, where $m/n \in \mathbb{Q}, (m, n) = 1$ and $S^1 = \{z \in \mathbb{C} \; |z|^2 < 1\}$ is a homomorphism of groups $(\mathbb{Q}, +)$ and $(S^1, \cdot)$. Find ker ϕ ,Im ϕ.

Practical no 2. Cayley's theorem and external direct product of groups

1. $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ has
   (a) 3 subgroups of order 2. (b) 7 subgroup of order 2 (c) 6 subgroups of order 2. (d) 9 subgroups of order 2.

2. The order of any non-identity element in $\mathbb{Z}_3 \times \mathbb{Z}_3$ is
   (a) 3 (b) 9 (c) 6 (d) none of these.

3. Which of the following statements is false?
   (a) $\mathbb{Z}_3 \times \mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_{15}$ (b) $\mathbb{Z}_3 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_6$
   (c) $\mathbb{Z}_9 \times \mathbb{Z}_9$ is isomorphic to $\mathbb{Z}_{27}$ (d) $\mathbb{Z}_4 \times \mathbb{Z}_3$ is isomorphic to $\mathbb{Z}_{12}$

4. The group $S_3 \times \mathbb{Z}_2$ is isomorphic to
   (a) $\mathbb{Z}_{12}$ (b) A_4 (c) D_6 (d) $\mathbb{Z}_6 \times \mathbb{Z}_2$

5. Let $G_1 = \mathbb{Z}_4 \times \mathbb{Z}_{15}$ and $G_2 = \mathbb{Z}_6 \times \mathbb{Z}_{10}$, then
   (a) G_1 and G_2 are cyclic groups of order 60.
   (b) G_1 and G_2 are not cyclic groups.
   (c) G_1 is cyclic but G_2 is not cyclic group.
   (d) G_1 is not cyclic but G_2 is a cyclic group.

6. . Which is true about groups?
   (a) $\mathbb{Z}_4 \times \mathbb{Z}_2$ is isomorphic to $V_4 \times \mathbb{Z}_2$.
   (b) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to $V_4 \times \mathbb{Z}_2$.
   (c) $V_4 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.
   (d) D_4 (the dihedral group of order 8) is isomorphic to Quaternion group Q_8 of order 8.

7. A group of order n is isomorphic to
   (a) a subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$. (b) a subgroup of A_n.
   (c) a subgroup of D_n. (d) a subgroup of $\mathbb{Z}_{2n}$

8. $\mathbb{Z}_3$ is isomorphic to the following subgroup of S_3
   (a) $<(12)>$. (b) $<(13)>$ (c) A_3 (d) S_3 itself.

9. A group of order 4 in which every element satisfies the equation $x^2 = e$ is isomorphic to
   (a) $\mathbb{Z}_2 \times \mathbb{Z}_2$. (b) μ_4, the group of forth roots of unity under multiplication.
   (c) $(\mathbb{Z}_4, +)$ (d) $\{\bar{1}, \bar{3}, \bar{7}, \bar{9}\}$.

10. The smallest positive integer n for which there are two non-isomorphic groups of order n equals.
    (a) 2 (b) 4 (c) 6 (d) 8

11. For each positive integer n,
(a) There is a cyclic group of order n. (b) There are two non-isomorphic groups of order n.
(c) There is a non-abelian group of order n. (d) The number of non-isomorphic groups of order n is equal to n

12. A non-cyclic group of order 6 is isomorphic to
(a) $\mathbb{Z}_3 \times \mathbb{Z}_2$ (b) μ_6, the group of sixth roots of unity under multiplication.
(c) $U(14) = \{\bar{1}, \bar{3}, \bar{5}, \bar{9}, \bar{11}, \bar{13}\}$. (d) S_3

13. Let $G_1 = \mathbb{Z}_3 \times \mathbb{Z}_5, G_2 = \mathbb{Z}_3 \times \mathbb{Z}_9$. Then

    (a) G_1 is isomorphic to $\mathbb{Z}_{15}$ and G_2 is isomorphic to $\mathbb{Z}_{27}$.

    (b) G_1 and G_2 are not isomorphic to $\mathbb{Z}_{15}$, $\mathbb{Z}_{27}$ respectively.

    (c) G_1 is not isomorphic to $\mathbb{Z}_{15}$ but G_2 is isomorphic to $\mathbb{Z}_{27}$

    (d) G_1 is isomorphic to $\mathbb{Z}_{15}$ but G_2 is not isomorphic to $\mathbb{Z}_{27}$

14. The number of elements of order 4 in $\mathbb{Z}_8 \times \mathbb{Z}_4$ is
(a) 4 (b) 8 (c) 20 (d) 16

15. Consider the following groups i) $\mathbb{Z}_4$ ii) $U(10)$ ii) $U(8)$ iv) $U(5)$. The only non-isomorphic group among them is
(a) $U(8)$ (b) $U(10)$ (c) $\mathbb{Z}_4$ (d) All are isomorphic.

16. Consider the following groups i) S_3 ii) μ_6 ii) $\mathbb{Z}_6$ iv) $\mathbb{Z}_2 \times \mathbb{Z}_3$ v) $U(9)$. The only non-isomorphic group among them is
(a) S_3 (b) μ_6 (c) $\mathbb{Z}_2 \times \mathbb{Z}_3$ (d) $S_3 \simeq U(9)$ and $\mu_6, \mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_3$ are isomorphic. .

17. If for positive integers m, n have $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to$(\mathbb{Z}_{mn}, +)$ then which is not true,

    (a) m, n are relatively prime.

    (b) m, n are odd.

    (c) m, n are prime.

    (d) $m = p^r, n = q^s$ for primes p, q and $r, s \in \mathbb{N}$.

18. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ and $H = \mathbb{Z}_4 \times \{\bar{0}, \bar{1}\}, K =< (\bar{1}, \bar{2}) >$ be subgroups of G Then
(a) G/H is isomorphic to G/K (b) G/H is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$
(c) H and K are isomorphic. (d) none of these.

19. From the given list of pairs of group, pick the pair of non-isomorphic groups
(a) $3\mathbb{Z}/12\mathbb{Z}$ and $\mathbb{Z}_4$ (b) $8\mathbb{Z}/48\mathbb{Z}$ and $\mathbb{Z}_6$
(c) $\mathbb{Z}_4$ and V_4 (d) $(\mathbb{Z} \times \mathbb{Z})/(2\mathbb{Z} \times 2\mathbb{Z})$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$

20. From the given list of pairs of groups, pick the pairs of isomorphic groups
(a) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$ (b) $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_2$
(c) D_4 and $\mathbb{Z}_4 \times \mathbb{Z}_2$ (d) $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $V_4 \times \mathbb{Z}_2$

21. If G, H, K are finite Abelian groups and $G \times K \cong H \times K$, then

    (a) $G = H$

(b) G need not be isomorphic to H

(c) $G \cong H$

(d) None of these.

22. If G is an Abelian group with order mn where $(m,n) = 1$. $G(m) = \{g \in G : g^m = e\}$ and $G(n) = \{g \in G : g^n = e\}$, then

    (a) $G = G(n) \cup G(m)$

    (b) $G \cong G(n) \times G(m)$

    (c) $G = G(m)G(m)$

    (d) None of these.

23. $m, n \in \mathbb{N}$ with $m|n$ and $H = \{\overline{k} \in U(n) : k \equiv 1 \bmod m\}$. Then

    (a) $U(n)/H \cong U(m)$

    (b) $U(n)/U(m) \cong H$.

    (c) $U(m) \cong H$.

    (d) None of these.

24. $U(16)/ < \overline{9} >$ is isomorphic to
    (a) $\mathbb{Z}_4$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\mathbb{Z}_8$ (d) Non of these. If $\mathbb{R}^*$ and $\mathbb{R}^+$ are multiplicative groups and $f : \mathbb{R}^* \to \mathbb{R}^+$ is defined by $f(x) = |x|$ then

    (a) f is an injective homomorphism.

    (b) f is not a homomorphism.

    (c) $\ker f = \{-1, 1\}$.

    (d) f is an isomorphism.

Practical 2 Descriptive Question

1. (a) Find all subgroup of order 2 in the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

    (b) Find all subgroups of order 4 in the group $\mathbb{Z}_4 \times \mathbb{Z}_4$.

    (c) Prove or disprove: $\mathbb{Z} \times \mathbb{Z}$ is a cyclic group.

2. (a) Find a subgroup of S_4 isomorphic to i) $\mathbb{Z}_4$ ii) V_4.

    (b) Find a subgroup of S_6 isomorphic to Z_6.

3. Find the left Cayley representation of S_3 in S_6.

4. Find the Cayley representation of $\mathbb{Z}_3$ in S_3.

5. Check whether ,

    (a) $\mathbb{Z}_3 \times \mathbb{Z}_9$ and $\mathbb{Z}_{27}$ are isomorphic groups.

    (b) $\mathbb{Z}_3 \times \mathbb{Z}_5$ and $\mathbb{Z}_{15}$ are isomorphic groups.

6. Show that $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ defined by $\phi(a, b) = a - b$ is a group homomorphism. Find Ker ϕ and describe the set $\phi^{-1}(3)$.

7. Let $G_1 \times G_2$, where $G_1 = (\mathbb{Z}_4, +), G_2 = \{\bar{1}, \bar{3}\}$ modulo 4 under multiplication. Let $H =< (\bar{2}, \bar{3}) > K =< (\bar{2}, \bar{1}) >$ be subgroups of G. List elements in H and $K, G/H$ and G/K. Show that H is isomorphic to K but G/H is not isomorphic to G/K.

8. Show that $\mathbb{Z}_8 \times \mathbb{Z}_4$ and $\mathbb{Z}_{8,00,000} \times \mathbb{Z}_{4,00,000}$ have same number of elements of order 4.

9. Find all subgroups of order 4 in $\mathbb{Z}_4 \times \mathbb{Z}_4$.

10. Find the number of elements of order 2 in $\mathbb{Z}_{2,00,000} \times \mathbb{Z}_{4,00,000}$.

11. Find a subgroup of $\mathbb{Z}_{12} \times \mathbb{Z}_4 \times \mathbb{Z}_{15}$ of order 9.

12. Let m, n be fixed positive integers. Consider the map $\phi_{m,n} : \mathbb{Z} \to \mathbb{Z}_m \times \mathbb{Z}_n$ defined by $\phi_{m,n}(x) = (x \; mod \; m, x \; mod \; n)$. Show that $\phi_{m,n}$ is a group homomorphism. Find ker $\phi_{m,n}$.

13. Show that a group G has a non-trivial homomorphism to $\mathbb{Z}_2$ if and only if G has a subgroup of index 2.

14. Let G be an Abelian group and $n \in \mathbb{N}$. $G(n) = \{g \in G : g^n = e\}$ and $G_n = \{g^n : g \in G\}$, then show that $G/G(n) \cong G_n$.

Practical no 3. Ring, Subring, Ideal and Integral domain

1. Let R be a ring and a, b be non-zero elements of R. The equation $ax = b$
   (a) has a unique solution in R.
   (b) may have more than one solution in R.
   (c) has atmost one solution in R.
   (d) None of these.

2. The group of units of the ring $\mathbb{Z}_{25}$ is
   (a) $\{\bar{1}, \bar{3}, \bar{5}, \bar{7}\}$ mod 25.
   (b) $\{\bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{6}, \bar{7}, \bar{8}, \bar{9}, \bar{11}, \bar{12}, \bar{13}, \bar{14}, \bar{16}, \bar{17}, \bar{18}, \bar{19}, \bar{21}, \bar{22}, \bar{23}, \bar{24}\}$ mod 25.
   (c) $\{\bar{1}, \bar{4}, \bar{8}, \bar{12}, \bar{16}, \bar{20}\}$ mod 25.
   (d) $\{\bar{1}, \bar{3}, \bar{6}, \bar{9}, \bar{12}, \bar{15}, \bar{18}, \bar{21}, \bar{24}\}$ mod 25.

3. The group of units of a ring is
   (a) abelian but may not be cyclic (b) Cyclic (c) may not be abelian (d) finite

4. Consider the ring $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ under addition and multiplication of 2×2 matrices, then $A \in M_2(\mathbb{Z})$ is an unit if -
   (a) $ad - bc \neq 0$
   (b) $ad - bc$ is an even integer.
   (c) and only if $ad - bc \neq 0$.
   (d) $ad - bc = \pm 1$.

5. Consider the following rings :
   (i) $(\mathbb{Z}_5, +, \cdot)$ (ii) $(\mathbb{Z}_{15}, +, \cdot)$ (iii) $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication (iv) $\mathbb{R}[x]$,Then
   (a) (i), (iv) have no proper zero divisors.
   (b) (i), (iii) have no proper zero divisors
   (c) (i), (iii) have proper zero divisors.
   (d) (i), (iii), (iv) have no proper zero divisors

6. The number of units in the ring $\mathbb{Z}_{20}$ is
   (a) 5 (b) 6 (c) 7 (d) 8

7. Which of the following is a subring of $(\mathbb{Q}, +, \cdot)$
   (i) $R = \{a/b \,/\, a, b \in \mathbb{Z},\ (a, b) = 1, b \neq 0, b$ is not divisible by $3\}$.
   (ii) $R = \{a/b \,/\, a, b \in \mathbb{Z},\ (a, b) = 1, b \neq 0, b$ is divisible by $3\}$.
   (iii) $R = \{a/b \,/\, a, b \in \mathbb{Z},\ (a, b) = 1, b \neq 0, a$ is divisible by $3\}$.
   (iii) $R = \{x^2 : x \in \mathbb{Q}\}$.
   (a) (i) and (iv) (b) (ii) and (iv) (c) (i) and (ii) (d) only (i).

8. Let R and S be rings. Consider the ring $R \times S$ under component wise addition and multiplication.
   (a) If R, S are integral domains, then $R \times S$ is an integral domain.
   (b) $R \times S$ is an integral domain if and only if R, S are integral domains.
   (c) $R \times S$ is not an integral domain, whatever R, S may be.
   (d) $R \times S$ is not commutative even if R, S are commutative.

9. Let R be an integral domain. Then, $x^2 = 1$
(a) has exactly two solutions. (b) may not have any solution.
(c) may have more than two solutions. (d) None of these.

10. Consider the following rings: (i) $\mathbb{Z}_{18}$ (ii) $\mathbb{Z}_{12}$ (iii) $\mathbb{Z}_{10}$ (iv) $\mathbb{Z}_{14}$, then
(a) (i), (ii), (iii) ,(iv) have nilpotent elements. (b) (i), (ii) have nilpotent elements.
(c) (iii), (iv) have nilpotent elements. (d) None of these have nilpotent elements.

11. In an integral domain the number of elements which are their own inverses is
(a) 1 (b) 1 or 2 (c) 2 (d) infinitely many.

12. In a ring $(\mathbb{Z}_n, +, \cdot)$ where n is a positive integer > 1

(i) $\bar{a}^2 = \bar{a} \;\Rightarrow\; \bar{a} = 0$ or $\bar{a} = \bar{1}$ for $\bar{a} \in \mathbb{Z}_n$.

(ii) $\bar{a} \cdot \bar{b} = \bar{0} \;\Rightarrow\; \bar{a} = \bar{0}$ or $\bar{b} = \bar{0}$ for $\bar{a}, \bar{b} \in \mathbb{Z}_n$.

(iii) $\bar{a} \cdot \bar{b} = \bar{a} \cdot \bar{c}$, $\bar{a} \neq 0 \;\Rightarrow\; \bar{b} = \bar{c}$ for $\bar{b}, \bar{c} \in \mathbb{Z}_n$. Then,

(a) the statements (i), (ii), (iii) are true.

(b) the statements (i) is true but (ii), (iii) may not be true.

(c) the statements (i), (ii), (iii) are true if n is prime.

(d) None of the above.

13. If R is a ring and a, b are zero divisors in R, then

(a) $a + b$ is always a zero divisor.

(c) $a + b$ may not be a zero divisor.

(b) $a + b$ is not a unit in R.

(d) None of these.

14. In the ring $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}_2 \right\}$, the number of non-zero zero divisors is
(a) 6 (b) 7 (c) 5 (d) None of these.

15. If x is an idempotent element in $\mathbb{Z}_n$ $(x^2 = x)$, then
(a) $1 - x$ is a unit. (b) $1 + x$ is a unit.
(c) $1 - x$ is an idempotent. (d) None of these.

16. Let R be a commutative ring such that $a^2 = 0 \Rightarrow a = 0 \;\forall\, a \in R$, then
(a) R has no proper zero divisors. (b) R has no nilpotent elements.
(c) R is an integral domain but not a field. (d) None of these.

17. Consider the rings $R_1 = (\mathbb{Z}_{10}, +, \cdot), R_2 = (\mathbb{Z}_{23}, +, \cdot), R_3 = M_2(\mathbb{Z})), R_4 = \mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
(a) R_1, R_2, R_3, R_4 are all integral domains. (b) Only R_2, R_3, R_4 are integral domains.
(c) R_2 is an integral domain. (d) R_2, R_4 are integral domains.

18. Let R be an integral domain of characteristic p. Then,
(a) $(x + y)^m = x^m + y^m \;\; \forall x, y \in R$ if and only if $m = p$.
(b) $(x + y)^m = x^m + y^m \;\; \forall x, y \in R$ and $m = kp$.
(c) $(x + y)^{p^n} = x^{p^n} + y^{p^n} \;\; \forall x, y \in R$ and for all $n \in \mathbb{N}$.
(d) None of the above.

19. Consider the subset $S = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}\}$ of $\mathbb{Z}_{10}$.
(a) S is a subring of $\mathbb{Z}_{10}$. (b) S is not a subring of $\mathbb{Z}_{10}$.
(c) S is a subring with multiplicative identity $\bar{6}$.
(d) S is a ring with multiplicative identity $\bar{6}$.

20. Let R be a ring in which $x^2 = x$ for all $x \in R$. Then,
(a) R is an integral domain with characteristic 3.
(b) R is field with characteristic 3.
(c) Characteristic of R is 2. (d) None of these.

21. The characteristics of the ring $\mathbb{Z}_{12} \times \mathbb{Z}_{15}$ under component wise addition and multiplication is
(a) 180 (b) 3 (c) 60 (d) 6

22. $x \in \mathbb{R}[x]$ is
   (a) is a unit in $\mathbb{R}[x]$.
   (b) is a zero divisor in $\mathbb{R}[x]$.
   (c) is neither a unit nor a zero divisor in $\mathbb{R}[x]$.
   (d) None of these.

23. If $E_{1,1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $R = M_{2\times 2}(\mathbb{R})$ then $RE_{1,1}$ is
   (a) is a subring with multiplicative identity I_2.
   (b) is a subring with multiplicative identity $E_{1,1}$.
   (c) is not a subring of R.
   (d) None of these.

24. Which of the following is true
   (a) $\mathbb{Z}_2[i], \mathbb{Z}_5[i]$ are integral domains and $\mathbb{Z}_3[i]$ is a field.
   (b) $\mathbb{Z}_2[i], \mathbb{Z}_5[i]$ and $\mathbb{Z}_3[i]$ are fields
   (c) $\mathbb{Z}_2[i], \mathbb{Z}_5[i]$ are fields and $\mathbb{Z}_3[i]$ is an integral domain.
   (d) Only $\mathbb{Z}_2[i]$ is a field and $\mathbb{Z}_3[i], \mathbb{Z}_5[i]$ are integral domains.

25. If $H_{\mathbb{Z}} = \{a + bi + cj + dk : a, b, c, d \in \mathbb{Z}\}$ then the multiplicative group of units of $H_{\mathbb{Z}}$ is
   (a) $\{\pm 1\}$.
   (b) $\{1, i, j, k\}$
   (c) $\{\pm 1, \pm i, \pm j, \pm k\}$
   (d) $H_{\mathbb{Z}} - \{0\}$.

Practical 3. Descriptive Questions

1. Let $(R, +, \cdot)$ be a ring with unity 1. Define $\bigoplus$ and $\bigodot$ on R as $a \bigoplus b = a + b - 1_R$ and $a \bigodot b = a + b - a \cdot b$. Show that $(R, \bigoplus, \bigodot)$ is a ring.

2. Let $(G, +)$ be an Abelian group and End $(G) = \{\Phi : G \to G : \Phi \text{ is homomorphism }\}$. For $\Phi_i \in \text{End}(G)$ define $(\Phi_1 + \Phi_2)(g) = \Phi_1(g) + \Phi_2(g)$ and $(\Phi_1 \cdot \Phi_2)(g) = \Phi_1 \circ \Phi_2(g)$. Show that End (G) is a ring.

3. (i) Let R be a ring such that $x^3 = x$ for all $x \in R$. Then show that R is commutative.
   (ii) Let R be a ring such that $x^4 = x$ for all $x \in R$. Then show that R is commutative.
   (iii) Let R be a ring in which $ab = ca \Rightarrow b = c$ for all $a, b, c \in R$, $a \neq 0$. Show that R is commutative.
   (iv) If R is a ring with more than one element . If $ax = b$ has a solution for all non-zero $a \in R$ and for all $b \in R$, then show that R is a division ring.

4. Show that $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication is a ring. Is it an integral domain? Justify.

5. Show that $R_p = \{m/n \ : \ m, n \in \mathbb{Z}; (m, n) = 1; p \nmid n\}$ for a fixed prime p is a ring.

6. Let X be a any set and $\mathbb{P}(X)$ be power set of X. For any $A, B \in \mathbb{P}(X)$, define $A + B = (A - B) \cup (B - A)$ and $A \cdot B = A \cap B$. Show that $(\mathbb{P}(X), +, \cdot)$ is a commutative ring.

7. Show that $\mathbb{Z}[i] = \{a + bi \ : \ a, b \in \mathbb{Z}\}$ is an integral domain.

8. (i) Show that a ring that is cyclic under addition is commutative.
   (ii) If R is a ring having 6 elements then R is commutative. Is R an integral domain? Justify.

9. Give examples -
   (a) of a finite ring which is non-commutative.
   (b) of a ring R such that $a^2 = a$ for all $a \in R$.

10. Let R be an integral domain and $a, b \in R$.
    (i) If $a^7 = b^7$, $a^{12} = b^{12}$, show that $a = b$.
    (ii) If $a^m = b^m$ and $a^n = b^n$ where m, n are coprime integers, then show that $a = b$.

11. Let $H = \left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \ : \ z, w \in \mathbb{C} \right\}$. Show that H is a non-commutative subring of $M_2(\mathbb{C})$ which is a division ring.

12. Show that, $R = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z}_7 \right\}$ under usual matrix addition and multiplication where addition and multiplication of entries is modulo 7 is a commutative ring. Is R an integral domain? Justify your answer. What happens if $R = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{Z}_5 \right\}$?

13. (i) Let R be a commutative ring. If u is a unit and a is nilpotent in R, show that $u + a$ is a unit in R.
    (ii) Let x be a non-zero element of a ring R. If there exists a unique $y \in R$ such that $xyx = x$, then show that x is invertible in R.

(iii) If a, b are nilpotent elements of a commutative ring, show that $a+b$ is also nilpotent. Give an example to show that this may fail if the ring is not commutative.

(iv) Determine all zero divisors, units and idempotent and nilpotent elements of the following rings- (a) $(\mathbb{Z}_{18}, +, \cdot)$. (b) $\mathbb{Z}_3 \times \mathbb{Z}_6$ under component wise addition and multiplication. (c) $F \times F$ where F is a field. (d) $(\mathbb{P}(X), +, \cap)$.

(v) Find two elements a, b in a ring such that both are non-zero zero divisors in R, $a+b \neq 0$ and $a+b$ is not a zero divisor.

14. (i) Let $a \neq 0$ be nilpotent in a commutative ring R, show that $(1-a)$ is unit in R.

(ii) Show that 0 is the only nilpotent element in an integral domain R. Prove that only idempotents in R are 0 and 1.

(iii) Find a zero divisor and a non-zero idempotent other that 1 in $\mathbb{Z}_5[i] = \{a + ib : a, b \in \mathbb{Z}_5[i]\}$.

(iv) If a is an idempotent in $\mathbb{Z}_n$, show that $1-a$ is also an idempotent.

15. In the following examples, check whether S is a subring of the given ring R.

(a) $R = M_2(\mathbb{R})$, $S = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \text{ s.t. } a + c = b + d \right\}$.

(b) $R = M_2(\mathbb{R})$, $S = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

(c) $R = M_2(\mathbb{Q})$, $S = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Q} \right\}$.

(d) $R = \mathbb{Q}$, $S = \{a/b : a, b \in \mathbb{Z}, (a, b) = 1, b \text{ is odd }\}$.

(e) $R = \mathbb{Q}$, $S = \{a/b : a, b \in \mathbb{Z}, (a, b) = 1, b \text{ is even }\}$.

16. Show that $\mathbb{Z}[\sqrt{2}]$ has infinitely many units. (every$(3+2\sqrt{2})^n$ is a unit where n is a positive integer.)

17. (i) Consider the ring $R = \{0, 2, 4, 6, 8, 10\}$ under addition and multiplication modulo 12. What is the characteristic of R?

(ii) Let R be a ring in which $x^4 = x$ for all $x \in R$. Find the characteristic of R.

(iii) For integral domain R with characteristic p, show that $(x+y)^p = x^p + y^p$ for every $x, y \in R$.

(iv) For integral domain R with characteristic p, show that $(x+y)^{p^n} = x^{p^n} + y^{p^n}$ for every $x, y \in R$ and positive integer n.

(v) Prove or disprove: A ring with characteristic n is finite.

18. Let R be an integral domain with characteristic 2. Show that -

(a) $(a+b)^2 = a^2 + b^2 \ \forall a, b \in R$.

(b) $S = \{a \in R \ a^2 = a\}$ is a subring of R.

19. Find elements x and y in a ring of characteristic 4 such that $(x+y)^4 \neq x^4 + y^4$.

Practical no 4. Homomorphism, Isomorphism of Rings

1. Consider the ring $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
Let $I = \{(a, -a) \ : \ a \in \mathbb{Z}\}, J = \{(a, 0) \ : \ a \in \mathbb{Z}\}$. Then,
(a) I and J are ideals of $\mathbb{Z} \times \mathbb{Z}$.
(b) I is an ideal of $\mathbb{Z} \times \mathbb{Z}$ but J is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.
(c) Neither I nor J is an ideal of $\mathbb{Z} \times \mathbb{Z}$.
(d) J is an ideal of $\mathbb{Z} \times \mathbb{Z}$ but I is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.

2. Consider the ring $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ and
let $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5 \right\}$. Then
   (a) I is a subring of $M_2(\mathbb{Z})$ but not an ideal.
   (b) I is an ideal of $M_2(\mathbb{Z})$ but not a subring.
   (c) I is not an ideal of $M_2(\mathbb{Z})$.
   (d) I is both a subring and an ideal of $M_2(\mathbb{Z})$.

3. Consider the ideal $I = 10\mathbb{Z}$ and $J = 12\mathbb{Z}$, then
(a) $I + J = 22\mathbb{Z}$, $IJ = 120\mathbb{Z}$. (b) $I + J = 2\mathbb{Z}$, $IJ = 60\mathbb{Z}$.
(c) $I + J = 2\mathbb{Z}$, $IJ = 120\mathbb{Z}$. (d) None of these.

4. In the ring of integers $\mathbb{Z}$, consider the ideals $I = 4\mathbb{Z} + 6\mathbb{Z}$, $J = m\mathbb{Z} + n\mathbb{Z}$, $m, n \in \mathbb{N}$. Then,
(a) $I = 24\mathbb{Z}, J = mn\mathbb{Z}$ (b) $I = 2\mathbb{Z}, J = d\mathbb{Z}$ where $d = \ gcd\ (m, n)$
(c) $I = 12\mathbb{Z}, J = l\mathbb{Z}$, where $l = \ lcm\ (m, n)$ (d) None of the above.

5. In the ring of integers $\mathbb{Z}$, consider the ideal $I = (6\mathbb{Z})(4\mathbb{Z})$, then
(a) $I = 24\mathbb{Z}$. (b) $I = 12\mathbb{Z}$. (c) $I = 2\mathbb{Z}$. (d) None of these.

6. $(n\mathbb{Z})(m\mathbb{Z}) = (n\mathbb{Z}) \cap (m\mathbb{Z})$ if and only if
(a) $m|n$. (b) $(m, n) = 1$. (c) $m = n$. (d) None of these.

7. Which of the following is true
(i) $\mathbb{Z}$ is an ideal of $\mathbb{Q}$ (ii) $\{(n, n) : n \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$
(iii) $\{f \in F(\mathbb{R}, \mathbb{R}) : f(\pi) = 0\}$ where $F(\mathbb{R}, \mathbb{R})$ is a ring of real valued functions.
(a) (i), (ii) (b) (ii), (iii) (c) only (iii). (d) None of these.

8. The number of ring homomorphisms from $\mathbb{Q}$ to itself is
(a) 1 (b) 2 (c) infinitely many (d) none of these.

9. The number of ring homomorphisms from $\mathbb{C}$ to itself is
(a) 1 (b) 2 (c) infinitely many (d) none of these.

10. Consider the following pair of rings.
   (i) $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$ (ii) $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-5}]$ (iii) $\mathbb{Q}$ and $\mathbb{R}$ (iv) $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$,$\mathbb{C}$

   (a) (i) and (iv) are isomorphic pairs of rings.
   (b) (i) and (ii) are isomorphic pairs of rings.

(c) only (iv) is an isomorphic pair of rings.

(d) (i), (ii) and (iv) are isomorphic pairs of rings.

11. Consider the following maps from $M_2(\mathbb{Z}_p) \to \mathbb{Z}_p$ defined by
$f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a$, $g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a + d$, $h(A) = det\, A$ for $A \in M_2(\mathbb{Z}_p)$.
(a) f, g, h are all ring homomorphisms. (b) only h is a ring homomorphism.
(c) f is a ring homomorphism and g, h are not. (d) none of these.

12. Consider the map $\pi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ and $\pi_2 : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $\pi_1(m, n) = m$, $\pi_2(m) = (m, 0)$, where $\mathbb{Z}\times\mathbb{Z}$ denotes ring with component wise addition and multiplication.
(a) π_1 and π_2 are ring homomorphisms.
(b) Both π_1, π_2 are not ring homomorphisms.
(c) π_1 is a ring homomorphism but π_2 is not a ring homomorphism.
(d) π_2 is a ring homomorphism but π_1 is not a ring homomorphism.

13. The number of ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are
(a) one (b) zero (c) two (d) infinitely many.

14. Let $\phi_n : \mathbb{Z}[x] \to \mathbb{Z}_n[x]$ be defined by $\phi(a_0 + a_1x + \cdots a_kx^k) = \bar{a_0} + \bar{a_1}x + \cdots \bar{a_k}x^k$, is a ring homomorphism only if
(a) n is a prime number. (b) n is a positive integer.
(c) n is an odd integer. (d) n is an even integer.

15. The kernel of the ring homomorphism $\phi : \mathbb{R}[x] \to \mathbb{C}$ defined by $\phi(f(x)) = f(2 + i)$ is
(a) $\{f(x) \in \mathbb{R}[x] : (x - 2)|f(x)\}$. (b) $\{f(x) \in \mathbb{R}[x] : (x^2 - 4x - 5)|f(x)\}$.
(c) $\{f(x) \in \mathbb{R}[x] : (x^2 - 4x + 2)|f(x)\}$. (d) $\{f(x) \in \mathbb{R}[x] : (x^2 - 4x + 5)|f(x)\}$

16. Consider the ring homomorphism $\phi : \mathbb{R}[x] \to \mathbb{R}$ defined by $\phi(a_0 + a_1x + \cdots + a_nx^n) = a_0 + a_1 + \cdots + a_n$. Then, the kernel ϕ is
(a) $\{f(x) \in \mathbb{R}[x] : f(1) = 1\}$. (b) $\{f(x) \in \mathbb{R}[x] : f(1) = 0\}$.
(c) $\{f(x) \in \mathbb{R}[x] : f(0) = 1\}$. (d) None of these.

17. Consider the ring homomorphism $\phi : \mathbb{R}[x] \to \mathbb{R}$ defined by $\phi(a_0 + a_1x + \cdots + a_nx^n) = \sum_{k=0}^{n}(-1)^k a_k$. Then, the kernel ϕ is
(a) $\{f(x) \in \mathbb{R}[x] : f(-1) = 0\}$. (b) $\{f(x) \in \mathbb{R}[x] : f(-1) = 1\}$.
(c) $\{f(x) \in \mathbb{R}[x] : f(-1) = -1\}$. (d) None of these.

18. Ring $H = \left\{\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z}\right\}$ is

(a) is not isomorphic to $\mathbb{Z}[\sqrt{2}]$.

(b) is isomorphic to $\mathbb{Z}[\sqrt{2}]$.

(c) is isomorphic to $\mathbb{Q}[\sqrt{2}]$.

(d) None of these.

19. Consider the maps $\phi_1 : M_2(\mathbb{Z}) \to \mathbb{Z}$ defined by $\phi_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a$ and $\phi_2 : R \to \mathbb{Z}$ defined by $\phi_1\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a$, where $R = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z}\right\}$. Then
(a) ϕ_1 is a ring homomorphism but ϕ_2 is not a ring homomorphism.
(b) Both ϕ_1, ϕ_2 are ring homomorphisms.
(c) ϕ_2 is a ring homomorphism but ϕ_1 is not a ring homomorphism.
(d) Both ϕ_1, ϕ_2 are not ring homomorphisms.

20. The question ring $\dfrac{\mathbb{Z}[i]}{(1+i)}$ is
(a) is an infinite ring. (b) a field having 2 elements.
(c) a ring having 4 elements. (d) a ring with proper zero-divisors.

Practical 4 Descriptive Question

1. Check whether following sets are ideals of the ring $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
(a) $I = \{(a, a) : a \in \mathbb{Z}\}$. (b) $I = \{(2a, 2b) : a, b \in \mathbb{Z}\}$.
(c) $I = \{(2a, 0) : a \in \mathbb{Z}\}$. (d) $I = \{(a, -a) : a \in \mathbb{Z}\}$.

2. Check which of the following are ideals of the polynomial ring $\mathbb{Z}[x]$.
(a) $I = \{f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x] : 3|a_0\}$.
(b) $I = \{f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x] : 3|a_2\}$.
(c) $I = \{f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x] : f(0) = 0\}$.
(d) $I = \{f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x] : \sum_{i=0}^{n} a_i = 0\}$.

3. (a) Let R be a commutative ring and $a \in R$ be non-zero. Show that, annihilator of a, $ann(a) = \{r \in R : ra = 0\}$ is an ideal of R.
(b) If A, B are ideals of a commutative ring R such that $R = A + B$, show that $A \cap B = AB$.
(c) If A, B are ideals of a commutative ring R such that $A \cap B = (0)$ then show that $ab = 0$ for $a \in A, b \in B$.

4. Let $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even }\}$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$.

5. Show that $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are even integers }\right\}$ is an ideal of $M_2(\mathbb{Z})$.

6. Show that $I = \left\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R}\right\}$ is an ideal of the ring $I = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}\right\}$.

7. Show that $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5\right\}$ is an ideal of $M_2(\mathbb{Z})$.

8. Is $I = \{4a + bi : a, b \in \mathbb{Z}\}$ an ideal of $\mathbb{Z}[i]$? Justify.

9. Let I be an ideal of ring R. Show that, $I^m = \left\{ \sum_{i=1}^{n} a_{i1}a_{i2}\cdots a_{im} \,:\, a_{ij} \in R, n \in \mathbb{N} \right\}$ is an ideal of R.

10. Find the characteristic of $\mathbb{Z}[i]/ < 2 + i >$.

11. Let R be a ring with prime characteristic p and $f : R \to R$ is defined by $f(a) = a^p$ for $a \in R$. Show that f is a ring homomorphism (called the Frobenius homomorphism). Is it an isomorphism?

12. Show that the following are isomorphic:

    (a) $H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ and $\mathbb{Z}[\sqrt{2}]$.

    (b) $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}$ and $\mathbb{C}$.

13. Let R be a ring and $a \in R$ be a unit. Show that the map from R into itself given by $x \mapsto axa^{-1}$ is a ring automorphism.

14. Suppose I and J are ideals of ring R so that $R = I + J$. Let $\phi : R \to R/I \times R/J$ be given by $\phi(r) = (r + I, r + J)$, then show that ϕ is a ring homomorphism with ker $\phi = I \cap J$. Hence or otherwise show that the rings $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ when $(n, m) = 1$.

15. Let $\phi : \mathbb{Z} \to R$ be given by $\phi(n) = n\dot{1}_R$. Show that ϕ is a ring homomorphism and Im ϕ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}_k$ for $k \in \mathbb{N}$.

16. Consider the maps $\phi_1 : \mathbb{R}[x] \to M_2(\mathbb{R})$ defined by $\phi(a_0 + a_1x + \cdots + a_nx^n) = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}$ and $\phi_2 : \mathbb{R}[x] \to M_3(\mathbb{R})$ defined by $\phi(a_0 + a_1x + \cdots + a_nx^n) = \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{pmatrix}$. Show that ϕ_1, ϕ_2 are ring homomorphisms and find their kernels.

17. Show that following pairs of rings are not isomorphic.

    (a) the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$

    (b) the rings $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-5}]$

    (c) $\mathbb{Z}_4 \times \mathbb{Z}_6$ and $\mathbb{Z}_{24}$.

    (d) $\mathbb{R}$ and $\mathbb{C}$.

18. Let R be a commutative ring and I be its ideal. Show that (i) $J_1 = \{x \in R : xa = 0 \;\forall\, a \in I\}$ (ii) $J_2 = \{x \in R : x^n \in I, \text{ for some } n \in \mathbb{N}\}$ are ideals of R.

19. Find all the ideals of $\mathbb{Z}/12\mathbb{Z}$ using the correspondence theorem.

20. Show that $\mathbb{Z}[i]/(2 + i)$ is finite field, where $(2 + i) = \{(2 + i)(m + in) : m + in \in \mathbb{Z}[i]\}$.

21. Show that following rings of order 4 are non-isomorphic-

    (a) $\mathbb{Z}_4$ (b) $\mathbb{Z}_z \times \mathbb{Z}_2$ (c) $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$ (d) $\mathbb{Z}_2[x]/(x^2 + x + 1)$

22. Show that $|\mathbb{Z}[i]/(z)| = n^2 + m^2$ where $z = n + mi \in \mathbb{Z}[i]$.
23. Show that aR is an ideal of R if and only if $ar = ra$ for all $r \in R$.
24. Let $\phi : \mathbb{R} \to M_{2\times 2}(\mathbb{R})$ be given by $\phi(r) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Show that ϕ is a ring homomorphism and $\phi(1)$ is the multiplicative identity of $\text{Im}\,\phi$ but not of $M_{2\times 2}(\mathbb{R})$.
25. Give example of a simple ring which is not a field.

Practical no 5. Divisibility, Prime ideals, Maximal ideals

1. Let $R = M_2(\mathbb{Z})$ and $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and are divisible by } 5 \right\}$
   (a) I is not an ideal. (b) I is a prime ideal but not a maximal ideal.
   (c) I is a maximal ideal. (d) I is an ideal but not a prime ideal.

2. Let R be a commutative ring. If (0) is the only maximal ideal in R, then
   (a) R is finite ring. (b) R is an integral domain, but not field.
   (c) R is a field. (d) None of the above.

3. The number of maximal ideals in $\mathbb{Z}_{16}$ are
   (a) 1. (b) 2. (c) 3. (d) 4.

4. Let $R = M_2(\mathbb{Z}_2)$ and $I = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A : A \in R \right\}$. Then
   (a) I is not an ideal. (b) I is a prime ideal but not a maximal ideal.
   (c) I is a maximal ideal. (d) I is an ideal but not a prime ideal.

5. Let $R = C[0, 1]$, the ring of continuous real valued functions on [0,1] under pointwise addition and multiplication, $I = \{f \in R : f(1/2) = 0\}$.
   (a) I is not an ideal. (b) I is a prime ideal but not a maximal ideal.
   (c) I is a maximal ideal. (d) I is an ideal but not a prime ideal.

6. In the polynomial ring $\mathbb{Z}[x]$, consider $I = \{f(x) : f(0) = 0\}$, then
   (a) I is an ideal. (b) I is prime ideal but not maximal ideal.
   (c) I is a maximal ideal. (d) I is ideal but neither prime ideal nor maximal.

7. If R is an integral domain and I is a proper ideal then

   (a) R/I is an integral domain. (b) R/I is a field.
   (c) R/I is finite (d) R/I may not be commutative.

8. Let R be a finite commutative ring. Then
   (a) R is a field. (b) (0) is the only proper ideal of R.
   (c) every prime ideal is maximal. (d) R is an integral domain.

9. Let $S = \{a + ib : a, b \in \mathbb{Z}, \text{ are divisible by 5 } \}$. Then,
   (a) S is not an ideal but is a subring of $\mathbb{Z}[i]$.
   (b) S is an ideal as well as subring of $\mathbb{Z}[i]$.
   (c) S is an ideal of $\mathbb{Z}[i]$.
   (d) None of these.

10. Let R be a commutative ring, and P_1 and P_2 are prime ideals of R, then
    (a) $P_1 \cup P_2$ and $P_1 \cap P_2$ both are prime ideals of R.
    (b) $P_1 \cap P_2$ is prime ideal of R always but $P_1 \cup P_2$ may not be.
    (c) If $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ then $P_1 \cap P_2$ is prime ideal of R.
    (d) None of the above.

11. Which of the following is irreducible in $\mathbb{Z}[\sqrt{5}]$
(a) $9+4\sqrt{5}$ (b) $1+\sqrt{5}$ (c) 5 (d) $4+\sqrt{5}$

12. In the ring $\mathbb{Q}[x]$, the principal ideal (x^2+bx+c) is a maximal ideal if
(a) $b=c=0$ (b) b^2-4c is not a square of a rational number.
(c) b^2-4c is a square of a rational number. (d) b^2-4c is an integer.

13. In the ring $\mathbb{Z}[x]$,

(a) (x) is a maximal ideal.

(b) (x) is a prime ideal which is not maximal.

(c) there is no maximal ideal in $\mathbb{Z}[x]$.

(d) (x) is not a prime ideal.

14. In the ring $\mathbb{Z}[\sqrt{5}]$
(a) $1+\sqrt{5}$ is irreducible but not prime. (b) $1+\sqrt{5}$ is prime
(c) $1+\sqrt{5}$ is not irreducible (d) $1+\sqrt{5}$ is a unit.

15. In the ring $\mathbb{Z}[\sqrt{-5}]$
(a) $1+\sqrt{-5}$ is not irreducible (b) $1+\sqrt{-5}$ is prime
(c) $1+\sqrt{-5}$ is irreducible but not prime (d) $1+\sqrt{-5}$ is a unit.

16. Consider the following pairs of elements in the given rings respectively. (i) $2+i$ and $1-2i$ in $\mathbb{Z}[i]$ (ii) $1-\sqrt{-5}$ and $7-3\sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$ (iii) 2 and $1+i$ in $\mathbb{Z}[i]$. Then
(a) (i) and (iii) are pairs of associates (b) (i) and (ii) are pairs of associates
(c) (i), (ii) and (iii) are pairs of associates. (d) only (iii) is a pair of associates.

17. Consider the following elements in $\mathbb{Z}[\sqrt{-5}]$ (i) $6+\sqrt{-5}$ (ii) 7 (iii) $2-3\sqrt{-5}$. Then
(a) (ii) and (iii) are irreducible and (i) is not irreducible
(b) (i) and (iii) are irreducible and (ii) is not irreducible
(c) (i),(ii) and (iii) are all irreducible (d) (i),(ii) and (iii) are all reducible.

18. Which of the following is true in $\mathbb{Z}[\sqrt{-5}]$
(a) $2+\sqrt{-5}$ is irreducible but not prime. (b) $2+\sqrt{-5}$ is prime.
(c) 3 is prime. (d) 4 is reducible.

19. The number of maximal ideals in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}$ is
(a) 1. (b) 3. (c) 6. (d) 9.

20. Which of the following is prime in $\mathbb{Z}[i]$,
(a) 2. (b) 5. (c) 17. (d) 3.

21. $a\pm ib$ is irreducible in $\mathbb{Z}[i]$ satisfying $a^2+b^2=p$ if
(a) p is a prime integer. (b) p is an odd integer.
(c) $p=2$ or a prime such that $p\equiv 1 \bmod 4$. (d) None of these.

22. The quotient ring $\dfrac{\mathbb{Z}[i]}{(1+i)}$

(a) an integral domain which is not a field.

(b) a field having 2 elements.

(c) a field having 4 elements.

(d) a ring with proper zero divisors.

23. The ring $\frac{\mathbb{R}[x]}{(x^4+1)}$ is
(a) an infinite integral domain. (b) an infinite field.
(c) a finite field. (d) None of these.

24. Consider the ring homomorphisms $f_1 : \mathbb{Z}[i] \to \mathbb{Z}_2$ defined by $f(a+bi) = (a-b)\, mod\, 2$ and $f_2 : \mathbb{Z}[i] \to \mathbb{Z}_5$ defined by $f(a+bi) = (a-2b)\, mod\, 5$. Then

(a) ker f_1 is a maximal ideal but ker f_2 is not a maximal ideal.

(b) ker f_2 is a maximal ideal but ker f_1 is not a maximal ideal.

(c) both ker f_1 and ker f_2 are not maximal ideals.

(d) both ker f_1 and ker f_2 are maximal ideals.

25. In the ring $\mathbb{R}[x]$ and $\mathbb{C}[x]$, consider the ideal $I = (x^2 - x + 2)$

(a) I is a maximal ideal in both $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

(b) I is a maximal ideal in $\mathbb{R}[x]$ but not $\mathbb{C}[x]$.

(c) I is a maximal ideal in $\mathbb{C}[x]$ but not in $\mathbb{R}[x]$.

(d) I is a not maximal ideal in both $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

Practical 5 Descriptive Question

1. Let R, S be commutative rings. And $f : R \to S$ be an onto ring homomorphism. Prove that

   (i) If P is a prime ideal in S, then $f^{-1}(P)$ is a prime ideal in R.

   (ii) If M is a maximal ideal in S, $f^{-1}(M)$ is a maximal ideal in R. Do the above results hold if f is not onto? Justify your answer.

2. Prove or disprove : If R, S be commutative rings. And $f : R \to S$ be an onto ring homomorphism.

   (i) If P is a prime ideal in R, then $f(P)$ is a prime ideal in S.

   (ii) If M is a maximal ideal in R, $f(M)$ is a maximal ideal in M.

3. For a commutative ring R, prove that

   (i) R is an integral domain iff $\{0\}$ is a prime ideal in R.

   (ii) R is a field iff $\{0\}$ is a maximal ideal in R.

4. Let R be a ring and I be an ideal of R. Let M be an ideal of R containing I, and let $\bar{M} = M/I$ be the corresponding ideal of R/I. Prove that M is maximal if and only if $\bar{M}$ is maximal.

5. Let $R = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$. Let $\phi : R \to \mathbb{Z}$ be defined by $\phi\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix} \right) = a - b$. Then, is ker ϕ a prime ideal? Is ker ϕ a maximal ideal? Justify.

6. Show that the following ideals are maximal in the indicated ring

   (a) $I = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}, a - b \text{ is even }\}$ in $\mathbb{Z}[\sqrt{-5}]$.
   (b) $< x^2 + 1 >$ in $\mathbb{R}[x]$.
   (c) $I = \{(3x, y) : x, y \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$.
   (d) $I = \{f \in R : f(0) = 0\}$ in the ring of continuous function from $\mathbb{R}$ to $\mathbb{R}$.
   (e) $(\sqrt{2})$ in $\mathbb{Z}[\sqrt{2}]$.
   (f) $I = \{a + bi : a\,mod\,2 = b\,mod\,2\}$ in $\mathbb{Z}[i]$.

7. Find the maximal ideals of the following rings a) $\mathbb{Z}_8$ b) $\mathbb{Z}_{10}$ c) $\mathbb{R} \to \mathbb{R}$ under component wise addi (d)$\mathbb{Z}_{24}$ (e) $\mathbb{Q}$ (f) $\mathbb{Z} \times \mathbb{Z}$.

8. Determine the maximal ideals of each of the following (a) $\mathbb{R}[x]/(x^2)$ (b) $\mathbb{R}[x]/(x^2 - 3x + 2)$
   (c) $\mathbb{R}[x]/(x^2 + x + 1)$ (d) $R = \{\frac{a}{b} : a, b \in \mathbb{Z},\ (a, b) = 1,\ a, b \text{ is odd }\}$

9. Is (2) a maximal ideal in $\mathbb{Z}[i]$? Justify your answer.

10. Show that the following ideals are prime ideal in the indicated ring
    (a) I is set of all polynomials all of whose coefficients are even in $\mathbb{Z}[x]$.
    (b) $I = \{f(x) : f(0) = 0\}$ in $\mathbb{Z}[x]$. Also show that I is not maximal ideal.
    (c) $I = \{(x, 0) : x \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$. Also show that I is not a maximal ideal.
    (d) $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ divisible by } 5 \right\}$ in $M_2(\mathbb{Z})$.
    (e) $(x^3 + x + 1)$ in $\mathbb{Z}_2[x]$.
    (f) $I = \{(3x, y) : x, y \in \mathbb{Z}\}$ in $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.
    (g) $(x^2 + 1)$ in $\mathbb{Z}[x]$. Also show it is not maximal.

11. Show that $\mathbb{Z}[x]$ has infinitely many maximal ideals.

12. Determine which of the following are prime ideals in $\mathbb{Z}[i]$?
    (i) (2) (ii) (3) (iii) $(2 + i)$ (iv) $(1 + i)$.

13. Show that $(2 + i)$ is a maximal ideal in $\mathbb{Z}[i]$. How many elements does $\mathbb{Z}[i]/(2 + i)$ has?

14. Consider the ring $\mathbb{Z}[\sqrt{d}]$, where d is not 1 and is not divisible by square of number. Define $N : \mathbb{Z}[\sqrt{d}] \to \mathbb{Z}^+$ as $N(a + b\sqrt{d}) = |a^2 - db^2|$. Show that
    (a) $N(x) = 0$ if and only if $x = 0$.
    (b) $N(xy) = N(x)N(y)$
    (c) $N(x) = 1$ if and only if x is unit.
    (d) x is irreducible if $N(x)$ is prime.

15. Show that $I = \{f(x) \in \mathbb{Z}[x] : 2|f(0)\}$ is not a principal ideal in $\mathbb{Z}[x]$.

16. Show that $2, 3, 1 + \sqrt{5}, 1 - \sqrt{5}$ are irreducible elements in $\mathbb{Z}[\sqrt{5}]$. Which elements among these are prime?

Practical no 6. Polynomial ring and Field

1. If $R_1 = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \,:\, a, b \in \mathbb{Z}\}, R_2 = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} \,:\, a, b \in \mathbb{Z}\}, R_3 = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} \,:\, a, b \in \mathbb{Q}\}, R_3 = \mathbb{Z}[i] = \{a + bi \,:\, a, b \in \mathbb{Z}\}$, then
(a) R_1, R_2, R_3, R_4 are integral domains which are not fields.
(b) R_1, R_2, R_4 are integral domains which are not fields and R_3 is a field.
(c) R_1, R_2, R_3, R_4 are all fields.
(d) None of the above.

2. Consider the ring $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \; a \in \mathbb{Q} \right\}$
(a) S is an integral domain which is not a field.
(b) S is a field with multiplicative identity $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
(c) S is a non-commutative ring. (d) None of these.

3. Let R and S be rings. Consider the ring $R \times S$ under component wise addition and multiplication.
(a) If R, S are fields, then $R \times S$ is a field.
(b) If R, S are integral domains, then $R \times S$ is an integral domain.
(c) $R \times S$ is not a field, whatever R, S may be.
(d) None of the above.

4. Let F_1 and F_2 be fields having 9 and 16 elements respectively. Then, the number of (non-trivial) ring homomorphism from F_1 to F_2 are
(a) One (b) zero (c) two (d) None of the above.

5. Consider the rings $R_1 = (\mathbb{Z}_{10}, +, \cdot), R_2 = (\mathbb{Z}_{17}, +, \cdot), R_3 = \{\frac{a}{b} : a, b \in \mathbb{Z},\, b \text{ is odd } \}$.
(a) R_1, R_2, R_3 are all fields. (b) Only R_1, R_2 are fields.
(c) Only R_2, R_3 are fields. (d) Only R_2 is a field.

6. Let $\mathbb{R}[x]/(2x)$. Then,
(a) R is a field. (b) R is an integral domain but not a field.
(c) R is not an integral domain. (d) R is a finite commutative ring.

7. There exist fields of
(a) 10 elements. (b) 7,8,9 elements. (c) 12 elements. (d) 6 elements.

8. The field of quotients of $\mathbb{Z}[\sqrt{2}]$ is
(a) $\mathbb{Q}[\sqrt{2}]$ (b) $\mathbb{R}$ (c) $\mathbb{Q}$ (d) $\mathbb{C}$.

9. The field of quotients of $\mathbb{Z}[i]$ is
(a) $\mathbb{Q}[i]$ (b) $\mathbb{R}$ (c) $\mathbb{C}$ (d) None of these.

10. Which of the following statements is true?
(a) R is ring $\Rightarrow R[x]$ is ring.
(b) R is a division ring $\Rightarrow$ $R[x]$ is division ring.
(c) R is field $\Rightarrow$ $R[x]$ is field.

(d) R is integral domain $\Rightarrow$ $R[x]$ is an integral domain.

11. The polynomial $f(x) = 2x^2 + 4$ is reducible over
(a) $\mathbb{Z}$ (b) $\mathbb{Q}$ (c) $\mathbb{R}$ (d) None.

12. Which of the following polynomials in $\mathbb{Z}[x]$ satisfy an Eisenstein criterion for irreducibility in $\mathbb{Q}$.
(i) $x^2 - 12$ (ii) $8x^3 + 6x^2 - 9x + 24$
(iii) $4x^{10} - 9x^3 + 24x - 18$ (iv) $2x^{10} - 25x^3 + 10x^2 - 30$
(a) All are irreducible. (b) (ii) and (iii) are irreducible.
(c) (ii), (iii) and (iv) are irreducible. (d) only (i) is true.

13. The polynomial $8x^3 - 6x + 1$ is
(a) reducible over $\mathbb{Z}$ (b) is reducible over $\mathbb{Q}$
(c) is irreducible over $\mathbb{Q}$ (d) is irreducible over $\mathbb{R}$.

14. Let $f(x) = x^2 - 2$, then
(a) $f(x)$ is reducible in $\mathbb{Q}[x]$.
(b) $f(x)$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{Q}[\sqrt{2}][x]$.
(c) $f(x)$ is reducible over $\mathbb{Q}$
(d) None of these.

15. Let $f(x) = x^2 - 2$, then
(a) $f(x)$ is reducible in $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$.
(b) $f(x)$ is irreducible in $\mathbb{Z}_3[x]$ but reducible in $\mathbb{Z}_5[x]$.
(c) $f(x)$ is reducible in $\mathbb{Z}_3[x]$ but irreducible in $\mathbb{Z}_5[x]$.
(d) $f(x)$ is irreducible in both $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$.

16. Let R be a commutative ring and $f(x)$ be a polynomial of degree n over R. Then the no. of roots of $f(x)$ in R is
(a) less than or equal to n. (b) equal to n
(c) strictly less than n (d) may be greater than n.

17. The polynomial $2x + 1$ is
(a) unit in $\mathbb{Z}_8[x]$ (b) zero divisor in $\mathbb{Z}_8[x]$ but not nilpotent.
(c) nilpotent in $\mathbb{Z}_8[x]$ (d) None of the above

18. Let $f(x) \in \mathbb{Z}[x]$. Which of the following is true?
(a) If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.
(b) $f(x)$ is reducible over $\mathbb{Q}$, but it may not be reducible over $\mathbb{Z}$.
(c) $f(x)$ is reducible over $\mathbb{Q}$.
(d) none of these.

19. Let $I = (x^2 + x + 1)$ in $\mathbb{Z}_n[x]$, $1 \leq n \leq 10$ Then, $\mathbb{Z}_n[x]/I$ is a field if
(a) $n \leq 5$ (b) $n = 2$ (c) $n = 3$ (d) $n = 7$

20. The polynomial x is irreducible in $\mathbb{Z}_n[x]$
(a) for each n (b) for $n \geq 3$ (c) iff n is prime (d) never.

21. The number of roots of the polynomial $x^{25} - 1$ in $\mathbb{Z}_{37}[x]$ is
(a) 25 (b) 5 (c) 24 (d) 1

22. Let $f(x) = x^3 - x^2 + 1$
   (a) $(f(x))$ is a maximal ideal in $\mathbb{Z}_2[x], \mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$.
   (b) $(f(x))$ is a maximal ideal in $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$ but not in $\mathbb{Z}_2[x]$.
   (c) $(f(x))$ is a maximal ideal in $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$ but not in $\mathbb{Z}_5[x]$.
   (d) None of the above

23. Let $f(x) = x^{10} + x^9 + \cdots + x + 1, g(x) = x^{11} + x^{10} + x^9 + \cdots + x + 1$. Then
   (a) $f(x), g(x)$ are both irreducible over $\mathbb{Z}[x]$.
   (b) $f(x), g(x)$ are both reducible over $\mathbb{Z}[x]$.
   (c) $f(x)$ is irreducible over $\mathbb{Z}[x]$, $g(x)$ is not.
   (d) $g(x)$ is irreducible over $\mathbb{Z}[x]$, $f(x)$ is not.

24. The polynomial $f(x) = x$ is
(a) irreducible over any ring R. (b) irreducible but not prime over any ring R.
(c) can be factored in some polynomial ring. (d) has no roots.

25. In $\mathbb{R}[x]$, Let $I = \{f(x) \in \mathbb{R}[x] : f(2) = f'(2) = f''(2) = 0\}$ and $J = \{f(x) \in \mathbb{R}[x] : f(2) = 0, f'(3) = 0\}$
(a) I, J are ideals in $\mathbb{R}[x]$. (b) I is an ideal, J is not.
(c) Neither I nor J is an ideal. (d) I is a prime ideal in $\mathbb{R}[x]$.

Practical 6 Descriptive Question

1. (a) Show that $\mathbb{Q}[\sqrt{2}] = \{r + s\sqrt{2} : r, s \in \mathbb{Q}\}$ and $\mathbb{Q}[i] = \{r + si : r, s \in \mathbb{Q}\}$ are fields.
   (b) Show that $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[i]$ are integral domains but not fields. Find their field of quotients.

2. Check if the fields $\mathbb{Q}[\sqrt{2}]$ and $\mathbb{Q}[\sqrt{5}]$ are isomorphic.

3. Show that $\mathbb{Z}[i]/(3)$ is a field.

4. Prove that the ring $\mathbb{Z}_2[x]/(x^3 + x + 1)$ is a field, but $\mathbb{Z}_3[x]/(x^3 + x + 1)$ is not a field.

5. List all the polynomials of degree 2 over (a) $\mathbb{Z}_2[x]$ (b) $\mathbb{Z}_3[x]$. Which of these are irreducible?

6. Determine which of the given polynomials are irreducible over $\mathbb{Q}$
   (a) $x^5 + 9x^2 + 12x^2 + 6$ (b) $x^4 + 3x^2 + 3$ (c) $x^4 + x + 1$
   (d) $x^5 + 5x^2 + 1$ (e) $x^3 - 5x + 10$ (f) $x^4 - 3x^2 + 9$
   (g) $2x^5 - 5x^4 + 5$ (h) $x^8 + 8$.

7. Show that $x^4 + 1$ is reducible over $\mathbb{Z}_p[x]$, where p is prime $p > 2$.

8. Show that $x^3 + ax^2 + bx + 1 \in \mathbb{Z}[x]$ is reducible over $\mathbb{Z}$ iff either $a = b$ or $a + b = -2$.

9. Let $I = \{f(x) \in \mathbb{R}[x] : f(2) = f'(2) = f''(2) = 0\}$. Show that I is a principal ideal in $\mathbb{R}[x]$ and find its generator.

10. Let $f(x) = x^{10} + x^9 + \cdots + x + 1$, $g(x) = x^{11} + x^{10} + \cdots + x + 1$. Determine whether $f(x), g(x)$ are irreducible over $\mathbb{Q}$.

11. If $p > 2$ is a prime, then show that $1 - x + x^2 - x^3 + \cdots + x^{p-1}$ is irreducible in $\mathbb{Q}[x]$.

12. Let p be a positive prime and let $n \in \mathbb{N}$, Then show that $p^{n^{th}}$ cyclotomic polynomial $\Phi_{p^n}(x) = 1 + x^{p^{n-1}} + x^{2p^{n-1}} + \cdots + x^{(p-1)p^{n-1}}$ is irreducible in $\mathbb{Q}[x]$.

13. Show that $x^n - p$ is irreducible over $\mathbb{Q}$ for each prime p.

14. Determine all ideals in $\mathbb{Z}[x]/(2, x^3 + 1)$ where $(2, x^3 + 1) = (2) + (x^3 + 1)$.

15. Let $R = \{a_0 + a_1x + \cdots + a_nx^n : a_0 \in \mathbb{Z}, a_i \in \mathbb{Q} \text{ for} i \geq 1\}$. Show that R is an integral domain. Find units and primes in R. Is x a prime in R.

16. Let $\mathbb{F}$ be a field. Show that a polynomial $f(x) \in \mathbb{F}[x]$ of degree n has at the most n distinct roots in $\mathbb{F}$.

17. Let $\mathbb{F}$ be a field and $f(x) \in \mathbb{F}[x]$ is such that $a \in \mathbb{F}$ is a root of $f(x)$ as well as the derivative of $f(x)$ with respect to x, then show that a has multiplicity at least two.

18. For any ring R, show that $\dfrac{R[x]}{< x >} \simeq R$.

19. Prove that $\dfrac{\mathbb{Q}[x]}{(x^2 - 2)}$ is a field.

20. Let p be a prime. Show that the number of monic quadratic reducible polynomials in $\mathbb{Z}_p[x]$ is $\dfrac{p(p+1)}{2}$. Determine the number of monic quadratic irreducible polynomials in $\mathbb{Z}_p[x]$.

21. Show that $\mathbb{R}[x]/(x^2 + 1)$ is isomorphic to $\mathbb{C}$.

22. Show that $x^2 + 1$ and $x^2 + x + 4$ are irreducible polynomials in $\mathbb{Z}_{11}[x]$. Show that $\dfrac{\mathbb{Z}_{11}[x]}{(x^2 + 1)}$ and $\dfrac{\mathbb{Z}_{11}[x]}{(x^2 + x + 4)}$ are fields having 121 elements.

23. Show that a finite field containing p^n elements where p is a prime integer has characteristic p.

24. Suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. If r is a rational such that $x - r$ divides $f(x)$, show that r is an integer.

25. let F be a field and let a be a non-zero element of F.
    (a) If $af(x)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$.
    (b) If $f(ax)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$.

(c) If $f(x+a)$ is irreducible in $F[x]$, prove that $f(x)$ is irreducible in $F[x]$.

(d) use part (c) to prove that $8x^3 - 6x + 1$ is irreducible over $\mathbb{Q}$.

26. If p is a prime, prove that $x^{p-1} - x^{p-2} + \cdots - x + 1$ is irreducible over $\mathbb{Q}$.

27. Let F be a field having 32 elements. Then show that the only sub field of F is $\{0, 1\}$ and F itself.

28. Construct a field of order (i) 25 (ii) 27.

29. Show that for every prime p there exists a field of order p^2.

30. For prime integer p show that $\mathbb{Z}_{p^n}$ has a unique maximal ideal.

Practical no 7. Unit-wise Theoretical Questions

Unit I

1. Let H be a subgroup of group G. Prove that the following statements are equivalent.
   (a) $aHa^{-1} \subseteq H$ for each $a \in G$.
   (b) $aHa^{-1} = H$ for each $a \in G$.
   (c) Every left coset of H in G is also a right coset of H in G i.e. $aH = Ha$ for each $a \in G$.
   (d) $HaHb = Hab$ for each $a, b \in G$.
2. Let G be a group. Show that centre of a G is a normal subgroup of G.
3. If H is a normal subgroup of G and K is a subgroup of G, Show that $HK = KH$.
4. Let G be a group and $a \in G$, Show that $N(a) = \{x \in G : ax = xa\}$ is a subgroup of G and $< a >$ is a normal subgroup of $N(a)$.
5. Let G be finite group with a normal subgroup H such that $(\circ(H), \circ(G/H)) = 1$ then show that H is a unique subgroup of G of order H.
6. Let G be a group and H is a unique subgroup of a given order, then show that H is a normal subgroup of G.
7. Let H and K be subgroup of a group G such that $H \cap K = \{e\}$ then show that $hk = kh,\ h \in H, k \in K$.
8. Let G be a group such that $(ab)^n = a^n b^n$ for some position integer n.
   (a) Show that $G(n) = \{x^n / x \in G\}$ is a normal subgroup of G.
   (b) Show that $G(n-1) = \{x^{n-1)} \, / \, x \in G\}$ is a normal subgroup of G.
9. Let H be a normal subgroup of G and let $\dfrac{G}{H} = \{Ha : a \in G\}$. Show that $HaHb = Hab$ is a well defined binary operation in $\dfrac{G}{H}$ and $\dfrac{G}{H}$ is a group under this binary operation.
10. Let G be a group and H be a subgroup of G. If $x^2 \in H$ for each $x \in G$, then show that H is a subgroup of G and G/H is Abelian.
11. If $G/Z(G)$ is cyclic then prove that G is Abelian.
12. If a cyclic subgroup H of a group G is normal in G. Show that every subgroup of H is normal in G .
13. Let G be a group and H be a normal subgroup of G. Then prove that
    (a) $(Ha)^n = Ha^n$ for all $n \in \mathbb{Z}$.
    (b) $\circ(Ha)$ divides $\circ(a)$.
14. Let G, G' be groups and $f : G \to G'$ be an onto homomorphism. Prove that

(a) kernel f is a normal subgroup of G and $Im f$ is a subgroup of G'.

(b) If H' is a subgroup of G' then $f^{-1}(H') = \{h \in H : f(h) \in H'\}$ is a subgroup of G containing $\ker f$. If H' is normal in G' then $f^{-1}(H')$ is normal in G.

(c) If H is a subgroup G then $f(H) = \{f(h) : h \in H\}$ is a subgroup of G' and $f(Ha) = f(H)f(a)$ for each $a \in G$. Further, if H is normal in G then $f(H)$ is normal in G'.

15. Let G be a group and H be a normal subgroup of G. Show that $\eta : G \to G/H$ defined by $\eta(a) = Ha$ is a group homomorphism and $Ker\,\eta = H$.

16. State and prove "First isomorphism theorem / Fundamental theorem of homomorphism of groups".

17. State and prove " Second isomorphism theorem of groups".

18. State and prove "Third isomorphism theorem of groups".

19. State and prove Cayley's theorem for finite group.

20. Show that A_n is a normal subgroup of S_n.

21. Show that

(i) finite cyclic group of order n is isomorphic to the group $\mathbb{Z}_n$ of residue classes modulo n.

(ii) Every infinite cyclic group is isomorphic to the group $\mathbb{Z}$ of integers under addition.

OR

(i) Show that any two cyclic groups of same order are isomorphic.

(ii) Show that any two infinite cyclic groups are isomorphic.

22. Classify groups of order ≤ 7 up to isomorphism.

(i) Show the there are two non-isomorphic groups of order 4.

(ii) Show that there are only two non-isomorphic groups of order 6.

23. If $(G_1, \cdot), (G_2, *)$ are groups and $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$ with binary operation $\circ$ defined by $(g_1, g_2) \circ (g_1', g_2') = (g_1 \cdot g_1', g_2 * g_2')$ then

(a) $(G_1 \times G_2, \circ)$ is a group.

(b) if G_1, G_2 are abelian then $G_1 \times G_2$ is also abelian.

(c) If $a \in G_1, b \in G_2$ such that $\circ(a) = m, \circ(b) = n$, then $(a, b)^k = (a^k, b^k)$ and $\circ(a, b) = lcm(m, n)$.

(d) If G_1, G_2 are cyclic then $G_1 \times G_2$ is cyclic if and only if $\circ(G_1)$ and $\circ(G_2)$ are relatively prime.

(e) If H_1, H_2 are normal subgroups of G_1, G_2 respectively then $H_1 \times H_2$ is a normal subgroup of $G_1 \times G_2$ and $\dfrac{G_1 \times G_2}{H_1 \times H_2}$ is isomorphic to the external direct product $\dfrac{G_1}{H_1} \times \dfrac{G_2}{H_2}$.

Unit II

1. R is a ring with multiplicative identity, then
   (a) Show that the set of units in R form a group under multiplication.
   (b) The set $Z(R) = \{a \in R \;:\; ax = xa;\ \forall\, x \in R\}$, called the center of the ring is a subring of R.

2. (a) Show that every element of a finite commutative ring is either a unit or a zero divisor.
   (b) Show that every element of $\mathbb{Z}_n$ is either a unit or a zero divisor.
   (c) Show that an integral domain has no non-zero nilpotent element.

3. Show that subring of an integral domain is an integral domain.

4. (a) Show that, characteristic of a ring R is n if and only if the order of the multiplicative identity of R is n in the group $(R, +)$.
   (b) Show that characteristic of an integral domain is either 0 or a prime.

5. (a) Let R be a ring with unity 1_R and I be an ideal in R such that $1_R \in I$ then prove that $I = R$.
   (b) Let R be a commutative ring and $a \in R$. Prove that $Ra = (a) = \{ra/r \in R\}$ is an ideal of R.
   (c) Show that any ideal of the ring $\mathbb{Z}$ is of the form $m\mathbb{Z}$ for some $m \in \mathbb{Z}$.

6. (a) If I is an ideal of a ring R, then show that $R/I = \{x + I \;:\; x \in R\}$ is a ring with the operations $(x + I) + (y + I) = (x + y) + I$ and $(x + I)(y + I) = xy + I$.
   (b) Let R be a commutative ring. If I, J are ideals in R, Show that $I \cap J$, $I + J$ and IJ are ideals of R, where
   $I + J = \{x + y \;:\; x \in I, y \in J\}$ and $IJ = \left\{ \sum_{i=1}^{n} x_i y_i \;:\; x_i \in I, y_i \in J, n \in \mathbb{N} \right\}$.
   (c) Let R be a ring and I, J, K be ideals of R. Prove (a) $I(J + K) = IJ + IK$, $(I + J)K = IJ + JK$. (b) If $J \subseteq I$, then $I \cap (J + K) = J + (I \cap K)$.
   (d) For a ring R, show that any ideal of the ring of $n \times n$ matrices over R, $M_n(R)$ is of the form $M_n(I) = \{[a_{ij}] \;:\; a_{ij} \in I\}$ for some ideal I of R.

7. Show that a commutative ring is a field if and only if it has no proper ideal.

8. Let I be an ideal in a ring R and $\eta : R \to R/I$ is defined by $\eta(a) = a + I$ for $a \in R$. Show that η is a homomorphism and $\ker\, \eta = I$.

9. Let R be a commutative ring. Show that $I = \{a : a \in R, a^n = 0 \text{ for some } n \in \mathbb{N}\}$ is an ideal (called the nil radical) of R and R/I has no nilpotent element.

10. Let R, R' be commutative rings and $f : R \to R'$ be a ring homomorphism. Show that-
    (a) If f is surjective, I is an ideal of R, then $f(I)$ is an ideal of R'.
    (b) If I' is an ideal of R' , then $f^{-1}(I')$ is an ideal of R.

11. State and prove the First Isomorphism Theorem(Fundamental theorem of homomorphism) of rings.

12. (Second Isomorphism Theorem of rings) Let A be a subring and B be an ideal of a ring R. Then $A \cap B$ is an ideal of A and $A/(A \cap B) \simeq (A + B)/B$.

13. (Third Isomorphism Theorem of rings) Let A, B be ideals of a ring R with $A \subseteq B$. Then A/B is an ideal of R/B and $(R/B)/(A/B) \simeq R/A$.

14. Show that, $\bar{J}$ is an ideal of the quotient ring R/I if and only if there is an ideal $J \subseteq I$ of the ring R such that $\bar{J} = \{x + I : x \in J\}$.

15. There is exactly one non-zero ring homomorphism from $\mathbb{Z}$ into any ring R.

16. Let $f : R \to S$ be an onto ring homomorphism and $K = ker\ f$. Prove that there is one-one onto correspondence between ideals of R containing K and ideals of S.

Unit III

Fields

1. Show that a field is an integral domain. Is the converse true? Justify your answer.

2. Show that a finite integral domain is a field. Give an example of an infinite integral domain which is not a field.

3. Show that characteristic of a field is either zero or a prime number.

4. Show that the ring $\mathbb{Z}_n$ of residue classes modulo n is field if and only if n is a prime number.

5. Show that a field has no ideals except 0 and itself.

6. Show that an ideal P in a commutative ring R is a prime ideal if and only if R/P is an integral domain.

7. Show that an ideal M in a commutative ring R is a maximal ideal if and only if R/M is a field.

8. (a) If R is a finite commutative ring prove that every prime ideal is maximal.
    (b) If R is a commutative ring such that for $a \in R$ there exists a $n \in \mathbb{N}$ (depending on a) such that $a^n = a$ then show that every prime ideal is maximal.

9. (a) Show that an ideal I in the ring $\mathbb{Z}$ of integers is a prime ideal if and only if $I = (0)$ or $I = p\mathbb{Z}$ where p is a prime number.

(b) Show that every non-zero prime ideal in $\mathbb{Z}$ is a maximal ideal.

(c) Show that an ideal I in the ring $\mathbb{Z}$ of integers is a maximal ideal if and only if $I = p\mathbb{Z}$ where p is a prime number.

10. Show that a field contains a subfield isomorphic to $\mathbb{Z}_p$ or $\mathbb{Q}$.

11. Explain construction of quotient field of $\mathbb{Z}$.

12. Show that the rings, $\mathbb{Z}[i], \mathbb{Z}[\sqrt{2}], \mathbb{Z}[\sqrt{-5}]$ are integral domain which are not fields. Show that their quotient fields are $\mathbb{Q}[i], \mathbb{Q}[\sqrt{2}], \mathbb{Q}[\sqrt{-5}]$ respective.

Polynomial Rings

1. Let R be a ring. Let $R[x] = \{a_n x^x + a_{n-1}x^{n-1} + \cdots a_1 x + a_0 : \; a_i \in R, n \in \mathbb{Z}^+\}$. Show that $R[x]$ is a ring with respect to usual addition and multiplication of polynomial. Further show that if R is an integral domain, then $R[x]$ is also an integral domain.

2.

3. Let $\mathbb{F}$ be a field.

   (a) Show that $\mathbb{F}[x]$ is an integral domain. Is it a field? Justify your answer.

   (b) Show that only units in $\mathbb{F}[x]$ are the non-zero elements of $\mathbb{F}$.

   (c) Division Algorithm: For any pair of non-constant polynomials $f(x), g(x) \in \mathbb{F}[x]$, there exist $q(x), r(x) \in \mathbb{F}[x]$ such that $f(x) = g(x)q(x) + r(x)$ where $r(x) = 0$ or $deg\, r(x) < deg\, g(x)$.

4. Let F be a field. Show that every ideal of $F[x]$ is principal ideal.

5. Let F be a field $a \in F$, and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only $x - a$ is a factor of $f(x)$.

6. Define irreducible polynomials. Let F be a field, $f(x) \in F[x]$ and deg $f(x) = 2$ or 3. Show that $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F.

7. Show that

   (a) if F is a field, $f(x)$ and $g(x)$ in $F[x]$ are associate if and only if $f(x) = cg(x)$ where $c \neq 0$ in R.

   OR

   if R is an integral domain $f(x)$ and $g(x)$ in $R[x]$ are associate iff $f(x) = cg(x)$ where c is a unit in R.

   (b) Let F be a field and let $f(x), g(x), h(x) \in F[x]$. If $f(x)$ is irreducible over F and $f(x)|g(x)h(x)$,then $f(x)|g(x)$ or $f(x)|h(x)$.
   (In $\mathbb{R}[x]$ or $\mathbb{Q}[x], \mathbb{C}[x]$) if f(x) is irreducible and $f(x)|g(x)h(x)$,then $f(x)|g(x)$ or $f(x)|h(x)$.

8. Let $\mathbb{F}$ be a field. Show that $(p(x))$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is an irreducible polynomial in $F[x]$.

OR

Let F be a field. Show that $F[x]/ < p(x) >$ is a field if and only if $p(x)$ is an irreducible polynomial in $F[x]$.

9. Show that any a non-zero ideal of $\mathbb{F}[x]$ is prime if and only if it is maximal.

10. Show that the only irreducible polynomials in $\mathbb{R}[x]$ are a linear polynomial $x-a$ or quadratic polynomial x^2+bx+c such that $b^2-4c<0$, where $a,b,c \in \mathbb{R}$.

OR

Show that the only maximal (or prime)ideals in $\mathbb{R}[x]$ are principal ideals $< x-a >$ or $< x^2+bx+c >$ such that $b^2-ac<0$, $a,b,c \in \mathbb{R}$.

11. Show that the only irreducible polynomials in $\mathbb{C}[x]$ are a linear polynomial $x-\alpha$ for $\alpha \in \mathbb{C}$.

OR

Show that the only maximal (or prime)ideals in $\mathbb{C}[x]$ are principal ideals $< x-\alpha >$ where $\alpha \in \mathbb{C}$.

12. **Eisenstein's Criteria for Irreducibility** Let $f(x) = x^n + +a_{n-1}x^{n-1} + \cdots a_1x + a_0 \in \mathbb{Z}[x]$. Let $p \in \mathbb{Z}$ be a prime such that $p \mid a_i$, for all $i = 0,1,2,\cdots n-1$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

13. Using Eisenstein's criteria show that the $p^{th}-$ Cyclotomic polynomial $\Phi_p(x) = x^{p-1}+x^{p-2}+\cdots+x+1$ where p is prime, is irreducible over $\mathbb{Q}$.

14. Let $f(x) = a_nx^n + \cdots + a_0 \in \mathbb{Z}[x]$ and $a_n \neq 0$ if $r/s \in \mathbb{Q}$,$(r,s) = 1$, $f(r/s) = 0$ then show that $r/a_n, s/a_0$.

15. Show that $p^{1/n}$ is irrational where $n > 1$ and p is a prime.

Divisibility

1. Let R be a commutative ring and $a,b,u \neq 0$. Then show that
   - (a) If u is an unit in R then $u|a$.
   - (b) $b \in (a) \Leftrightarrow a|b \leftrightarrow (b) \subseteq (a)$.
   - (c) a and b are associates $\leftrightarrow$ $(a) = (b)$
   - (d) If $a|1_R \leftrightarrow a$ is a unit and $R = (a)$.

2. Let R be an Integral Domain , Let $p \in R$. Then ,
   - (a) p is prime iff (p) is a non zero prime ideal of R.
   - (b) If p is prime then p is irreducible. Show that the converse is not true.

3. Prove that in $\mathbb{Z}$ (ring of integers) a non zero non unit element p is irreducible iff p is prime.

4. Let R be an Integral Domain and $a \in R$, $a \neq 0_R$. If (a) is maximal then a is irreducible. Give an example to show that converse is not true.

5. Let R be a commutative ring and I, J be prime ideals of R. Show that, $I \cap J$ is prime only if $I \subseteq J$ or $J \subseteq I$.

6. Let R be commutative and I, J be ideal of R and P is a prime ideal of R that contains $I \cap J$. Prove that either $I \subseteq P$ or $J \subseteq P$.