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Paper - I (Unit IV)

Order statistics

Def^n Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a p.d.f. $f(x)$ and c.d.f $F(x)$

$$[F(x) = P(X \le x)]$$

y 2, 3, 3

	X_1	X_2	X_3	X_4		$X_{(1)}$	$X_{(2)}$	$X_{(3)}$	$X_{(4)}$
1	2	4	3	6	$\to$	2	3	4	6
2	7	8	1	4	$\to$	1	4	7	8
3	9	5	6	2	$\to$	2	5	6	9

$$X_{(1)} \quad X_{(2)} \quad X_{(3)} \quad X_{(4)} \quad \ldots \quad X_{(n)}$$

If the sample observations are arranged in increasing (ascending) order, it is defined as Order Statistics to a corresponding r.s. of size n.

We derote he

$$X_{(1)} \le X_{(2)} \le X_{(3)} \le \cdots \le X_{(n)}$$

Thm. 1 Let $F_{X_{(k)}}(x)$ is the cdf of $X_{(k)}$

$f_{X_{(k)}}(x)$ $E(X)$

$$F_{X_{(k)}}(x) = \sum_{i=k}^{n} \binom{n}{i} [F(x)]^{i} \ldots{}^{n-i}$$

$f_X(x)$ $f(x)$ $F(x)$ $\quad F_{X_{(k)}}(x) = \sum_{i=k} \binom{n}{i} \ldots [1-F(x)]^{n-i}$

Pf: $X_1 \cdots X_n$ r.s. from pdf $f(x)$ & cdf $F(x)$

let Y = a r.v. that counts the number of $X_1 \cdots X_n \leq x$

1 $\quad 20 \quad 18 \quad 6 \quad 5 \quad 15 \quad$ 2

2 $\quad 19 \quad 16 \quad 11 \quad 8 \quad 4 \quad$ 1

$x = 8 \quad 78$

Y follows Binomial Dist.

$$\left[X \sim B(n, p) \quad p = P(\text{success}) \right.$$
$$\binom{n}{x} p^x q^{n-x}$$
$$\left. = \binom{n}{x} p^x (1-p)^{n-x} \right]$$

$$p = \cancel{P(X \leq x)} = F(x)$$
$$q = 1 - P(X > x) = 1 - F(x)$$
$$Y \sim B(n, F(x))$$

$X_1 \cdots X_n$, $F(x)$, $F_{X_{(k)}}(x)$, $X_{(1)} \cdots X_{(n)}$

$$LHS = F_{X_{(k)}}(x) = P[X_{(k)} \leq x]$$

$[X_1 \cdots X_n$ $\quad f(x) \quad F(x)$ $\quad X_{(1)} \; X_{(2)} \cdots X_{(n)}$ $\quad Y \sim B(n, F(x))$

$f(x) \quad F(x) \longrightarrow Y \sim B(n, F(x))$

$f(x_{(k)}) \quad F_{X_{(k)}}(x)$

Ex. $X_1 \cdots X_n$

$X_{(1)} \; X_{(2)} \cdots X_{(n)}$

$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$

$X_{(1)} \le X_{(2)} \le \cdots \le X_{(k)} \le X_{(k+1)} \le \cdots \le X_{(n)}$

$2 \le 5 \le \cdots \le \underset{X_{(k)}}{10} \le 12 \le 14 \cdots \le 20$

$x = 14$

$P(X_{(k)} \le x) = P(X_{(k)} \le 14)$

$Y = \le 14 \qquad =$

X_1	X_2	X_3	X_4	X_5
18	15	19	10	12
$X_{(1)}$	$X_{(2)}$	$X_{(3)}$	$X_{(4)}$	$X_{(5)}$
10	12	15	18	19

$x = 18.5$

$Y =$ count of no. of obs. $\le 18.5 = 4$

$P(Y \ge k)$

$[F_{X_{(k)}}(x)] = P[X_{(k)} \le 18.5]$

$= P[Y \ge k]$

$= P[\quad]$

$y = 4$

$$LHS = P[X_{(k)} \le x] \quad ??$$

$Y \sim B(n, F(x))$

$\binom{n}{x} p^x \, \xi^{n-x}$

$$= P[Y \geq k]$$

$$= \sum_{i=k}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i}$$

$$= RHS \quad \text{Proved}$$

② Thm. 2 $\quad f_{X_{(k)}}(x) = ?$

$X_1 \; X_2 \; \cdots \; X_n$ $\qquad f(x) \quad F(x)$

$X_{(1)} \; X_{(2)} \; \cdots \; X_{(n)}$ $\qquad f_{X_{(k)}}(x)$

$F_{X_{(k)}}(x)$

$$f_{X_{(k)}}(x) = \frac{d}{dx} F_{X_{(k)}}(x)$$

$$= \frac{d}{dx} \sum_{i=k}^{n} \binom{n}{i} [F(x)]^i [1-F(x)]^{n-i}$$

$$= \sum_{i=k}^{n} \binom{n}{i} \left[\frac{d}{dx} \underbrace{(F(x))^i}_{u} \underbrace{(1-F(x))^{n-i}}_{v} \right]$$

$$= \sum_{i=k}^{n} \binom{n}{i} \left[(F(x))^i \frac{d}{dx}(1-F(x))^{n-i} + (1-F(x))^{n-i} \frac{d}{dx}(F(x))^i \right]$$

$$= \sum_{i=k} \binom{n}{i} \Big[\ldots \frac{d}{dx} + (1-F(x)) \frac{d}{dx}(F(x)) \Big]$$

$$\left[\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \right]$$

$$\left[\frac{d}{dx}(1-F(x))^{n-i} = \frac{d}{d(1-F(x))}(1-F(x))^{n-i} \times \frac{d(1-F(x))}{dx} \right.$$

$$\left(\frac{dx^n}{dx} = nx^{n-1} \right)$$

$$= (n-i)(1-F(x))^{n-i-1} \left(0 - \frac{dF(x)}{dx}\right)$$

$$\frac{dF(x)}{dx} = f(x)$$

$$= (n-i)(1-F(x))^{n-i-1}(-f(x))$$

$$\frac{dF_{X_{(k)}}(x)}{dx} = f_{X_{(k)}}(x)$$

$$\frac{d}{dx}[F(x)]^i = \frac{d[F(x)]^i}{dF(x)} \times \frac{dF(x)}{dx}$$

$$\left. = i[F(x)]^{i-1} f(x) \right]$$

$$f_{X_{(k)}}(x) = \sum_{i=k}^{n} \binom{n}{i} \Big[[F(x)]^i \{(n-i)(1-F(x))^{n-i-1} f(x)\}$$

$$+ (1-F(x))^{n-i}\, i\, [F(x)]^{i-1} f(x) \Big]$$

$$= \sum_{i=k}^{n} \binom{n}{i} i [F(x)]^{i-1} (1-F(x))^{n-i} f(x) - \sum_{i=k}^{n} \binom{n}{i} (n-i) [F(x)]^i (1-F(x))^{n-i-1} f(x)$$

$$= f(x) \sum_{i=k}^{n} \frac{n!}{(i-1)!(n-i)!} (F(x))^{i-1} (1-F(x))^{n-i}$$

$$(F(x))^i (1-F(x))^{n-i-1}$$

$$n \quad n!$$

$$-f(x)\sum_{i=k}^{n}\frac{n!}{i!\,(n-i-1)!}(F(x))^{i}(1-F(x))$$

$$=f(x)\frac{n!}{(k-1)!\,(n-k)!}\left[F(x)\right]^{k-1}(1-F(x))^{n-k}$$

$$+\ f(x)\sum_{i=k+1}^{n}\frac{n!}{(i-1)!\,(n-i)!}(F(x))^{i-1}(1-F(x))^{n-i}$$

$$-\ f(x)\sum_{i=k}^{n}\frac{n!}{i!\,(n-i-1)!}(F(x))^{i}(1-F(x))^{n-i-1}$$

$$f_{X_{(k)}}(x)=f(x)\frac{n!}{(k-1)!\,(n-k)!}(F(x))^{k-1}(1-F(x))^{n-k}$$

Thm. 3. Cum. prob. dist. fn of $X_{(1)}$ & $X_{(n)}$

F

$X_1 \;\ldots\; X_n$

$X_{(1)}\le X_{(2)}\le \cdots \le X_{(n)}$

pf.

$$P\left[X_{(1)}>x\right]=P(X_1>x,\ X_2>x,\ \ldots\ X_n>x)$$

X_1	X_2	X_3	X_4
2	6	3	4

$X_{(1)}$	$X_{(2)}$	$X_{(3)}$	$X_{(4)}$
2	3	4	6

$x=1$

$X_{(1)}>1$

$P(X_i\le x)$
$F(x)$

$$=\prod_{i=1}^{n}P(X_i>x)$$

$$\prod^{n}\big(1-P(X_i\le x)\big)$$

$F(x)$

$= \prod_{i=1}^{n} 2 = 2^n$

$$= \prod_{i=1}^{n} \left(1 - P(X_i \le x)\right)$$

$$= \prod_{i=1}^{n} \left(1 - F(x)\right)$$

$$= \left[1 - F(x)\right)^n$$

$$\Rightarrow P(X_{(1)} > x) = \left[1 - F(x)\right]^n$$

$$\Rightarrow F_{X_{(1)}}(x) = P(X_{(1)} \le x) = 1 - P(X_{(1)} >)$$

$$= 1 - \left[1 - F(x)\right]^n \quad \checkmark$$

$$F_{X_{(n)}}(x) = ?$$

$$F_{X_{(n)}}(x) = P(X_{(n)} \le x)$$

$$= P(X_1 \le x, X_2 \le x \ldots X_n \le x)$$

$$= \prod_{i=1}^{n} P(X_i \le x)$$

$$= \prod_{i=1}^{n} F(x)$$

$$F_{X_{(n)}}(x) = \left[F(x)\right]^n \quad \checkmark$$

$$F_{X_{(1)}}(x) = 1 - \cancel{\prod_{i=1}^{n}} \left(1 - F(x)\right)^n$$

$$= 1 - \left(1 - F(x)\right)^n$$

$$= 1 - (1 - F(x))$$

Thm 4 Prob. dist. f^n of $X_{(1)}$ & $X_{(n)}$

$f(x_{(1)}) = ?$ $f[x_{(n)}) = ?$

$$F_{X_{(1)}}(x) = 1 - (1 - F(x))^n$$

$$f_{X_{(1)}} = \frac{d}{dx}\left[1 - (1 - F(x))^n\right]$$

$$= \frac{d(1)}{dx} - \frac{d(1 - F(x))^n}{dx}$$

$$= 0 - \frac{d(1 - F(x))^n}{d(1 - F(x))} \frac{d(1 - F(x))}{dx}$$

$\frac{dx^n}{dx} = nx^{n-1}$

$$= 0 - n(1 - F(x))^{n-1}\left(0 - \frac{dF(x)}{dx}\right)$$

$$f_{X_{(1)}} = n(1 - F(x))^{n-1} f(x) \checkmark$$

$$f_{X_{(n)}} = \frac{dF_{X_{(n)}}(x)}{dx}$$

$$= \frac{d}{dx}[F(x)]^n$$

$$= \frac{d[F(x)]^n}{F(x)} \frac{dF(x)}{dx}$$

$$= n(F(x))^{n-1} f(x) \checkmark$$

① $X_1, X_2 \dots X_n \sim Exp(\lambda)$

$f_{X_{(1)}}(x)$, $f_{X_{(n)}}(x)$

① $X_1, X_2, \ldots, X_n \sim Exp(\lambda)$

Find $f_{X_{(k)}}(x)$, $f_{X_{(1)}}(x)$, $f_{X_{(n)}}(x)$

Sol. $f_{X_{(k)}}(x) = \frac{n!\, f(x)}{(k-1)!\,(n-k)!} (F(x))^{k-1} (1-F(x))^{n-k}$

$f_{X_{(1)}}(x) = n(1-F(x))^{n-1} f(x)$

$f_{X_{(n)}}(x) = n(F(x))^{n-1} f(x)$

$f(x) = \lambda e^{-\lambda x}$; $x \geq 0$

$F(x) = P(X \leq x) = \int_0^x \lambda e^{-\lambda x} dx$

$= \lambda \frac{e^{-\lambda x}}{-\lambda} \Big|_0^x$

$= -(e^{-\lambda x} - 1)$

$= 1 - e^{-\lambda x}$

$f_{X_{(k)}}(x) = \frac{n!\,(\lambda e^{-\lambda x})}{(k-1)!\,(n-k)!} (1-e^{-\lambda x})^{k-1} (e^{-\lambda x})^{n-k}$ ✓

$f_{X_{(1)}}(x) = n(e^{-\lambda x})^{n-1} (\lambda e^{-\lambda x})$

$f_{X_{(n)}}(x) = n(1-e^{-\lambda x})^{n-1} (\lambda e^{-\lambda x})$

$f_{X_{(k)}} = \frac{n!}{(k-1)!\,(n-k)!} [F(x)]^{k-1} [1-F(x)]^{n-k} f(x)$

$\ldots\, [F(x)]^{i} [1-F(x)]^{n-i}$

$(k-1)!\,(n-k)!$

2. $F_{X(k)} = \sum_{i=k}^{n} \binom{n}{i} [F(x)]^{i} [1-F(x)]$

3. $f_{X(1)} = n(1-F(x))^{n-1} f(x)$

4. $f_{X(n)} = n[F(x)]^{n-1} f(x)$

<u>Problems</u> 2. $f(x) = 1 \;\; ; \; 0 < x < 1$

Find $f_{X(1)}$, $f_{X(n)}$, $f_{X(3)}$, $F_{X(3)}$

sample size $= 10$ $(n = 10)$

<u>Sol.</u> $f(x) = 1$, $F(x) = \int_0^x 1 \, dx$

$$= x \Big|_0^x = x$$

$$f_{X(1)} = n(1-F(x))^{n-1} f(x)$$
$$= 10(1-x)^{10-1} \times 1$$
$$= 10(1-x)^{9}$$

$$f_{X(n)} = n[F(x)]^{n-1} f(x)$$
$$= 10(x)^{10-1} \times 1 = 10x^{9}$$

$f_{X(3)}$, $F_{X(3)}$ $= ?$

$$f_{X(k)} = \frac{n!}{(k-1)!\,(n-k)!} (F(x))^{k-1} (1-F(x))^{n-k} f(x)$$

$$f_{X_{(k)}} = \frac{n!}{(k-1)!\,(n-k)!}[\;\;]\;f(x)$$

$$= \frac{10!}{\;\;}\;{}^{3-1}\left(1-x\right)^{10-3}$$

$$n-i$$

$$F_{X_{(k)}} =$$

$$= {}^{10}\binom{10}{i} x^{i}\left(1-x\right)$$ ✓

(3) $f(x) = 2x \;;\; 0 < x < 1$

No. of obs. $= n = 4$

Find $f(X_{(1)})$, $f(X_{(n)})$, $f(X_{(3)})$

$F(X_{(3)})$, $F(X_{(1)})$, $F(X_{(n)})$

Sol. $f(x) = 2x \quad 0 < x < 1$

$$F(x) = \int_0^x 2x\,dx = x^2$$

$$f_{X_{(1)}} = n\left(1 - F(x)\right)^{n-1} f(x)$$

$$= 4\left(1 - x^2\right)^{4-1} \times 2x$$

$$= 8x\left(1 - x^2\right)^3 \quad ; 0 < x < 1$$

$$F_{X_{(1)}} = 1 - \left(1 - F(x)\right)^n_4$$

$$= 1 - (1 - x^2)^1 \checkmark$$

$$f_{X_{(n)}} = n\left[F(x)\right]^{n-1} f(x)$$

$$= 4(x^2)^{4-1} \times 2x$$

$$= 4x^6 \times 2x$$

$$= 8x^7$$

$$F_{X_{(n)}} = \left[F(x)\right]^n$$

$$= (x^2)^4 = x^8$$

$$f_{X_{(k)}} = \frac{n!}{(k-1)!\,(n-k)!}\left[F(x)\right]^{k-1}\left[1 - F(x)\right]^{n-k} f(x)$$

$$f_{X_{(3)}} = \frac{4!}{2!\;1!}(x^2)^2 (1 - x^2)^1 \times 2x$$

$$= 24x^5 (1 - x^2)$$

$$F_{X_{(k)}} = \sum_{i=k}^{n} \binom{n}{i} \left[F(x)\right]^i (1 - F(x))^{n-i}$$

$$F_{X_{(3)}} = \sum_{i=3}^{4} \binom{4}{i} (x^2)^i (1 - x^2)^{4-i}$$

$$= \binom{4}{3}(x^2)^3 (1 - x^2)^1 + \binom{4}{4}(x^2)^4 (1 - x^2)^{4-4}$$

$$= \binom{4}{3} (x^2 (1-x) + \binom{4}{4} \ldots$$

$$= 4x^6(1-x^2) + x^8$$ ✓

Theorem joint distribution of two order statistics $X_{(r)}$ & $X_{(s)}$

sol.

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}$$

$$Y_1 \le Y_2 \le \cdots \le Y_n$$

$$f(x_{(r)}, x_{(s)}) = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!} f(x) f(y) [F(x)]^{r-1} [1-F(y)]^{n-s} [F(y)-F(x)]^{s-r-1}$$

$r \le s$

Let X1,X2, X3be a random sample from a distribution of the continuous type having pdf f(x)=2x, 0<x<1, zero elsewhere.

(a) compute the probability that the smallest of X1,X2, X3exceeds the median of the distribution.

(b) If Y1≤Y2≤Y3are the order statistics, find the correlation between Y2and Y3

$X_{(2)}$ $X_{(3)}$

sol.

$$X_{(1)} = Y_1 \quad f_{X_{(1)}} =$$

$$F_{X_{(1)}} =$$

$$P[X_{(1)} \ge Me] = ?$$

$$P(X \geq m_e) = P(X \leq m_e) = \frac{1}{2}$$

$$f(x) = 2x \quad ; 0 < x < 1$$

$$P(X \leq m_e) = \frac{1}{2}$$

$$= \int_0^{m_e} f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \int_0^{m_e} 2x\,dx = \frac{1}{2}$$

$$\Rightarrow \cancel{2}\frac{x^2}{\cancel{2}}\Big|_0^{m_e} = \frac{1}{2}$$

$$\Rightarrow m_e^2 - 0 = \frac{1}{2}$$

$$\Rightarrow m_e = \sqrt{1/2} = 0.707$$

$$f_{X_{(1)}} = n(1-F(x))^{n-1} f(x)$$

$$f(x) = 2x \quad ; \quad F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x 2x\,dx$$

$$= \cancel{2}\frac{x^2}{\cancel{2}}\Big|_0^x$$

$$= x^2$$

$$F(x) = x^2$$

$$f_{X_{(1)}} = n(1-F(x))^{n-1} f(x)$$

$$= 3(1-x^2)^2 \times 2x \quad ; 0 < x < 1$$

$$= 6x(1-x^2)^2$$

$$P[X_{(1)} \geq me] = ?$$

$$= P[X_{(1)} \geq 0.707]$$

$$= \int_{0.707}^{1} 6x(1-x^2)^2 \, dx$$

$$= \int_{0.707}^{1} 6x(1 - 2x^2 + x^4) \, dx$$

$$= 6 \int_{0.707}^{1} (x - 2x^3 + x^5) \, dx$$

$$= 6 \left[\frac{x^2}{2} - \frac{2x^4}{4} + \frac{x^6}{6} \right]_{0.707}^{1}$$

$$= 3x^2 - 3x^4 + x^6 \Big]_{0.707}^{1}$$

$$= (3 - 3 + 1) - \left(3(0.707)^2 - 3(0.707)^4 + (0.707)^6\right)$$

$$= 0.125$$

$$Corr(X_{(2)}, X_{(3)}) = ?$$

$$Corr(X_{(2)}, X_{(3)}) = \frac{Cov(X_{(2)}, X_{(3)})}{\sqrt{V(X_{(2)})} \sqrt{V(X_{(3)})}}$$

$$Cov\left(X_{(2)}, X_{(3)}\right) = E\left(X_{(2)}X_{(3)}\right) - E\left(X_{(2)}\right)E\left(X_{(3)}\right)$$

$$V\left(X_{(2)}\right) = E\left(X_{(2)}\right)^2 - \left[E\left(X_{(2)}\right)\right]^2$$

$$V\left(X_{(3)}\right) = E\left(X_{(3)}\right)^2 - \left[E\left(X_{(3)}\right)\right]^2$$

Hint

$$f\left(X_{(2)}\right) \rightarrow E\left(X_{(2)}\right) \rightarrow E\left(X_{(2)}\right)^2$$

$$f\left(X_{(3)}\right) \rightarrow E\left(X_{(3)}\right) \rightarrow E\left(X_{(3)}\right)^2$$

$$f\left(X_{(2)}, X_{(3)}\right) = ?$$

$$E\left(X_{(2)}X_{(3)}\right) = ?$$

$$f(x) = 2x \;;\; 0 < x < 1$$

$$f(2) = 2\times 2 \qquad f(0) = 2\times 0 \qquad f(y) = 2y \qquad f(2) = 22$$

$$f\left(X_{(2)}, X_{(3)}\right) = \frac{n!\; f(x)f(y)}{(r-1)!\,(s-r-1)!\,(n-s)!}\left[F(x)\right]^{r-1}\left[1-F(y)\right]^{n-s}\left[F(y)-F(x)\right]^{s-r-1}$$

$n=3$
$r=2$
$s=3$

$$= \frac{3!\; 2x \times 2y}{1!\;0!\;0!}\left(x^2\right)^1\left[1-y^2\right]^0\left[y^2-x^2\right]^0$$

$$0 \le x \le y \le 1$$

$$= 24xy \times x^2 \qquad ;\; 0 \le x \le y \le 1$$

$$= 24x^3y \qquad 0 \le x \le y \le 1$$

$$E\left[X_{(2)}X_{(3)}\right] = E(xy)$$

$$= \int_0^1\int_0^y xy \times 24x^3y \, dx\, dy$$

$$= \int_0 \int_0 xy \times 24 x y \, dx \, dy$$

$$\left[\int_0^1 \left(\int_x^1 xy \times 24x^3 y \, dy \right) dx \right]$$

$$= 24 \int_0^1 \left(\int_0^y x^4 y^2 \, dx \right) dy$$

$$= 24 \int_0^1 y^2 \left. \frac{x^5}{5} \right|_0^y dy$$

$$= \frac{24}{5} \int_0^1 y^2 \times (y^5 - 0) \, dy$$

$$= \frac{24}{5} \int_0^1 y^7 \, dy$$

$$= \frac{24}{5} \left. \frac{y^8}{8} \right|_0^1$$

$$E(X_{(2)} X_{(3)}) = \frac{24}{40} = \frac{3}{5} = 0.6$$

$$E(X_{(2)}), \quad E(X_{(3)}), \quad V(X_{(2)}) \quad V(X_{(3)})$$

$$f(X_{(r)}) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1 - F(x)]^{n-r} f(x)$$

$X_{(2)} = x$
$X_{(3)} = y$

$$f(X_{(2)}) = \frac{3!}{1!\,1!} [x^2]^1 [1 - x^2]^1 \times 2x$$

$$= 12 x^3 (1 - x^2) \quad 0 \le x \le 1$$

$$E(X_{(2)}) = E(X) = 12 \int_0^1 x \times x^3 (1 - x^2) \, dx$$

$$= 12 \left[\int_0^1 x^4 \, dx - \int_0^1 x^6 \, dx \right]$$

$$= 12\left[\int_0^1 x^4\,dx - \int_0^1 x^6\,dx\right]$$

$$= 12\left[\frac{x^5}{5}\Big|_0^1 - \frac{x^7}{7}\Big|_0^1\right]$$

$$= 12\left[\frac{1}{5} - \frac{1}{7}\right]$$

$$= 0.68857$$

$$E(X_{(2)}^2) = \int_0^1 x^2 \times 12x^3(1-x^2)\,dx$$

$$= 12\left[\int_0^1 x^5\,dx - \int_0^1 x^7\,dx\right]$$

$$= 12\left[\frac{x^6}{6}\Big|_0^1 - \frac{x^8}{8}\Big|_0^1\right]$$

$$= 12\left[\frac{1}{6} - \frac{1}{8}\right]$$

$$E(X_{(2)}^2) = 0.5$$

$$V(X_{(2)}) = E(X_{(2)}^2) - \left[E(X_{(2)})\right]^2$$

$$= 0.5 - (0.68)^2 = 0.037$$

$$f(X_{(3)}) = \frac{3!}{(3-1)!\,(3-3)!}(x^2)^{3-1}(1-x^2)^{3-3} \times 2x$$

$$= \frac{3!}{2!\times 1!}\,x^4 \times 2x \qquad 0 \le x \le 1$$

$$= 6x^5 \qquad 0 \le x \le 1$$

$$= 6x^{\cdot} \qquad 0 = \cdot$$

$$E(X_{(3)}) = \int_0^1 x \times 6x^5 \, dx$$

$$= 6 \frac{x^7}{7} \Big|_0^1 = 6/7 = 0.857$$

$$E(X_{(3)}^2) = \int_0^1 x^2 \times 6x^5 \, dx$$

$$= 6 \frac{x^8}{8} = 6/8$$

$$V(X_{(3)}) = E(X_{(3)}^2) - [E(X_{(3)})]^2$$

$$= \frac{6}{8} - \left(\frac{6}{7}\right)^2 = \frac{6}{392}$$

$$= 0.015$$

$$\mathrm{Corr}(X_{(2)}, X_{(3)})$$

$$= \frac{E(X_{(2)} X_{(3)}) - E(X_{(2)}) E(X_{(3)})}{\sqrt{V(X_{(2)})} \sqrt{V(X_{(3)})}}$$

$$= \frac{0.6 - 0.6857 \times 0.857}{\sqrt{0.037} \times \sqrt{0.015}}$$

$$= \frac{0.01124}{0.19235 \times 0.12247}$$

0.19235×0.1[illegible]

$= 0.53$