Practical - 6 -Objectives

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Practical - 06

Practical no 6. Polynomial ring and Field

1. If $R_1 = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}, R_2 = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}, R_3 = \mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}, R_3 = \mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$, then
   (a) R_1, R_2, R_3, R_4 are integral domains which are not fields.
   (b) R_1, R_2, R_4 are integral domains which are not fields and R_3 is a field.
   (c) R_1, R_2, R_3, R_4 are all fields.
   (d) None of the above.

Solution : −Option (b) as R_1 , R_2, R_4 are integral domain but R_3 is a field .

2. Consider the ring $S = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} \ a \in \mathbb{Q} \right\}$
   (a) S is an integral domain which is not a field.
   (b) S is a field with multiplicative identity $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$.
   (c) S is a non-commutative ring. (d) None of these.

Solution : −Option (b) S is a field and multiplicative identity is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$

AS , $\begin{pmatrix} a & a \\ a & a \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{a}{2}+\frac{a}{2} & \frac{a}{2}+\frac{a}{2} \\ \frac{a}{2}+\frac{a}{2} & \frac{a}{2}+\frac{a}{2} \end{pmatrix} = \begin{pmatrix} a & a \\ a & a \end{pmatrix}$

Hence , $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ is the multiplicative identity of S .

3. Let R and S be rings. Consider the ring $R \times S$ under component wise addition and multiplication.
   (a) If R, S are fields, then $R \times S$ is a field.
   (b) If R, S are integral domains, then $R \times S$ is an integral domain.
   (c) $R \times S$ is not a field, whatever R, S may be.
   (d) None of the above.

Solution : −Option (c) R × S is not a field what ever may be R and S .

Let us prove that $R \times S$ is not a field if R and S are any two non-zero rings. The following steps lead to a solution:

(1) Note that $(1_R, 0_S) \in R \times S$ where 1_R is the multiplicative identity of R and 0_S is the additive identity of S.

Let us prove that $R \times S$ is not a field if R and S are any two non-zero rings. The following steps lead to a solution:

(1) Note that $(1_R, 0_S) \in R \times S$ where 1_R is the multiplicative identity of R and 0_S is the additive identity of S.

(2) If T is a ring, if 0_T is the additive identity of T, and if $t \in T$, prove that $t0_T = 0_T = 0_T t$. (Hint: use the distributive law and write $0_T = 0_T + 0_T$.)

(3) Prove that $(1_R, 1_S)$ is the multiplicative identity of $R \times S$ where 1_R and 1_S are the multiplicative identities of R and S, respectively.

(4) If $(r, s) \in R \times S$, prove that $(r, s) \cdot (1_R, 0_S) = (r, 0)$.

(5) Finally, prove that there does not exist $(r, s) \in R \times S$ such that $(r, s) \cdot (1_R, 0_S) = (1_R, 1_S)$. Deduce that the element $(1_R, 0_S) \in R \times S$ has no multiplicative inverse. (Hint: note carefully where you use the fact that R and S are non-zero rings.)

5. Consider the rings $R_1 = (\mathbb{Z}_{10}, +, \cdot), R_2 = (\mathbb{Z}_{17}, +, \cdot), R_3 = \{\frac{a}{b} : a, b \in \mathbb{Z},\ b \text{ is odd }\}$.
   (a) R_1, R_2, R_3 are all fields. (b) Only R_1, R_2 are fields.
   (c) Only R_2, R_3 are fields. (d) Only R_2 is a field.

Solution : −Option (d) Only R_2 is a field as 17 is a prime number.

6. Let $\mathbb{R}[x]/(2x)$. Then,
   (a) R is a field. (b) R is an integral domain but not a field.
   (c) R is not an integral domain. (d) R is a finite commutative ring.

Solution : −Option (a)

$2x = 2 \times x$ as 2 is a unit and only divisor of 2x is 2 and x so (2x) is irreducible polynomial in R[x] so (2x) is maximal ideal in R[x] , so R[x] / (2x) is a field.

8. The field of quotients of $\mathbb{Z}[\sqrt{2}]$ is
   (a) $\mathbb{Q}[\sqrt{2}]$ (b) $\mathbb{R}$ (c) $\mathbb{Q}$ (d) $\mathbb{C}$.

Solution : −Option (a)$\mathbb{Q}(\sqrt{2})$

9. The field of quotients of $\mathbb{Z}[i]$ is
   (a) $\mathbb{Q}[i]$ (b) $\mathbb{R}$ (c) $\mathbb{C}$ (d) None of these.

Solution : −Option (a) $\mathbb{Q}(i)$

10. Which of the following statements is true?

(a) R is ring $\Rightarrow R[x]$ is ring.

(b) R is a division ring $\Rightarrow R[x]$ is division ring.

(c) R is field $\Rightarrow R[x]$ is field.

(d) R is integral domain $\Rightarrow R[x]$ is an integral domain.

Solution : −Option (c) R is a field then R[x] is also a field .

11. The polynomial $f(x) = 2x^2 + 4$ is reducible over

(a) $\mathbb{Z}$ (b) $\mathbb{Q}$ (c) $\mathbb{R}$ (d) None.

Solution : −Option (c) :

$f(x) = 2x^2 + 4 = 2\left(x + \sqrt{2}\right)\left(x - \sqrt{2}\right)$

f(x) is reducible in $\mathbb{R}$ as $\sqrt{2} \in \mathbb{R}$

12. Which of the following polynomials in $\mathbb{Z}[x]$ satisfy an Eisenstein criterion for irreducibility in $\mathbb{Q}$.

(i) $x^2 - 12$ (ii) $8x^3 + 6x^2 - 9x + 24$

(iii) $4x^{10} - 9x^3 + 24x - 18$ (iv) $2x^{10} - 25x^3 + 10x^2 - 30$

(a) All are irreducible. (b) (ii) and (iii) are irreducible.

(c) (ii), (iii) and (iv) are irreducible. (d) only (i) is true.

Solution : − Option (c)(i), (ii), (iv)are irreducible.

Eisenstein's Criteria for Irreducibility Let $f(x) = x^n + +a_{n-1}x^{n-1} + \cdots a_1x + a_0 \in \mathbb{Z}[x]$. Let $p \in \mathbb{Z}$ be a prime such that $p \mid a_i$, for all $i = 0, 1, 2, \cdots n-1$, $p \nmid a_n$ and $p^2 \nmid a_0$. Then $f(x)$ is irreducible in $\mathbb{Q}[x]$.

(i) f(x) = $x^2 + 0x - 12$

2 ∤ 1 , 2 ∤ 12 , 2∤ 0 , But $2^2 = 4 / 12$

3 ∤ 1 , 3 | 12 , 3 | 0 , But $3^2 = 9 \nmid 12$ so $x^2 - 12$ *is irreducible* $\mathbb{Q}[x]$.

(ii) f(x) = $8x^3 + 6x^2 - 9x + 24$

3 ∤ 8 , 3 | 6 , 3 | 9 , 3 | 24 But $3^2 = 9 \nmid 24$ so (ii) *is irreducible in* $\mathbb{Q}[x]$.

(iii) f(x) = $4x^{10} - 9x^3 + 24x - 18$

3 ∤ 4 , 3|0 , 3 | 9 , 3 | 24 , 3 | 18 But $3^2 = 9 \mid 18$ so (iii) *is not irreducible in* $\mathbb{Q}[x]$.

(iv) f(x) = $2x^{10} - 5x^3 + 10x^2 - 30$

5 ∤ 2 , 5|0 , 5 | 5 , 5 | 10 , 5 | 30 , But $5^2 = 25 \nmid 30$ so (iv) *is irreducible in* $\mathbb{Q}[x]$.

13. The polynomial $8x^3 - 6x + 1$ is

(a) reducible over $\mathbb{Z}$ (b) is reducible over $\mathbb{Q}$

(c) is irreducible over $\mathbb{Q}$ (d) is irreducible over $\mathbb{R}$.

Solution : $-$(b)

($8x^3 - 6x + 1$) is reducible over $\mathbb{Q}$.

write left as $f(x) = 8x^3 - 6x + 1 = g(2x)$, where $g(t) = t^3 - 3t + 1$.

let t=p+q, we get

$g(t) = g(p+q) = p^3 + 3p^2q + 3pq^2 + q^3 - 3(p+q) + 1$

$= (p^3 + q^3 + 1) + (3p^2q + 3pq^2 - 3p - 3q)$

$= (p^3 + q^3 + 1) + 3(p+q)(pq-1)$

suppose $p^3 + q^3 + 1 = 0$ and $pq - 1 = 0$, we find that

$p^3 + q^3 = -1$ and $p^3q^3 = 1$,

so, they are solutions of equation $y^2 + y + 1 = 0$. i.e. $e^{i2\pi/3}$ and $e^{-i2\pi/3}$.

where, $e^{i\theta} = cos(\theta) + i \times sin(\theta)$.

then, p, q are $e^{i2\pi/9}$ and $e^{-i2\pi/9}$, and

$t = p + q = e^{i2\pi/9} + e^{-i2\pi/9} = 2cos(2\pi/9)$.

so, $x = cos(2\pi/9)$ is a solution of $8x^3 - 6x + 1 = 0$.

So , x = 0.76604444311 is the solution , so it is reducible over $\mathbb{Q}$.

14. Let $f(x) = x^2 - 2$, then

(a) $f(x)$ is reducible in $\mathbb{Q}[x]$.

(b) $f(x)$ is irreducible in $\mathbb{Q}[x]$ but reducible in $\mathbb{Q}[\sqrt{2}][x]$.

(c) $f(x)$ is reducible over $\mathbb{Q}$

(d) None of these.

Solution : $-$

$f(x) = x^2 - 2$

So if we take , f(x) = 0 then we get

$x^2 - 2 = 0$

$\Rightarrow x^2 = 2 \quad \Rightarrow x = \pm\sqrt{2}$

So , ($x + \sqrt{2}$) and (x - $\sqrt{2}$) is the factor of f(x) .

But , $\sqrt{2} \notin \mathbb{Q}$, *but* $\sqrt{2} \in \mathbb{Q}\left[\sqrt{2}\right]$

So , f(x) is reducible in $\mathbb{Q}\left[\sqrt{2}\right][x]$ but not in $\mathbb{Q}[x]$

Option (b)

15. Let $f(x) = x^2 - 2$, then

(a) $f(x)$ is reducible in $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$.

(b) $f(x)$ is irreducible in $\mathbb{Z}_3[x]$ but reducible in $\mathbb{Z}_5[x]$.

(c) $f(x)$ is reducible in $\mathbb{Z}_3[x]$ but irreducible in $\mathbb{Z}_5[x]$.

(d) $f(x)$ is irreducible in both $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$.

Solution $: -$

Let us consider ,

$f(x) = x^2 - 2$, then

In $\mathbb{Z}_3 = \{ 0, 1, 2\}$,

$f(0) = 0^2 - 2 = -2$

$f(1) = 1^2 - 2 = -1$

$f(2) = 2^2 - 2 = 2$

So not reducible in $\mathbb{Z}_3[x]$

In , $\mathbb{Z}_5 = \{ 0, 1, 2, 3, 4\}$

$f(0) = 0^2 - 2 = -2$

$f(1) = 1^2 - 2 = -1$

$f(2) = 2^2 - 2 = 2$

$f(3) = 3^2 - 2 = 7 \equiv 2$

$f(4) = 4^2 - 2 = 14 \equiv 4$

So not reducible in $\mathbb{Z}_3[x]$

Option (d)

16. Let R be a commutative ring and $f(x)$ be a polynomial of degree n over R. Then the no. of roots of $f(x)$ in R is

(a) less than or equal to n. (b) equal to n

(c) strictly less than n (d) may be greater than n.

Solution $: -$ Option (a)

Let R be a commutative ring and f(x) be a polynomial of degree n over R. Then number of roots of f(x) will be less than equal to n.

17. The polynomial $2x + 1$ is

(a) unit in $\mathbb{Z}_8[x]$ (b) zero divisor in $\mathbb{Z}_8[x]$ but not nilpotent.

(c) nilpotent in $\mathbb{Z}_8[x]$ (d) None of the above

Solution $: -$

$\mathbb{Z}_8 = \{ 0, 1, 2, 3, 4, 5, 6, 7\}$

When , $x = 0$ we get , $2x + 1 = 2 \times 0 + 1 = 1$

$x = 1$, $2x + 1 = 2 \times 1 + 1 = 3$

$x = 2$, $2x + 1 = 2 \times 2 + 1 = 5$

$x = 3$, $2x + 1 = 2 \times 3 + 1 = 7$

$x = 4$, $2x + 1 = 2 \times 4 + 1 = 9 = 1$

x = 5 , 2x + 1 = 2 × 5 + 1 = 11 = 3

Hence , Option (a) units of $\mathbb{Z}_8[x]$.

18. Let $f(x) \in \mathbb{Z}[x]$. Which of the following is true?

(a) If $f(x)$ is reducible over $\mathbb{Q}$, then it is reducible over $\mathbb{Z}$.

(b) $f(x)$ is reducible over $\mathbb{Q}$, but it may not be reducible over $\mathbb{Z}$.

(c) $f(x)$ is reducible over $\mathbb{Q}$.

(d) none of these.

Solution : −(**b**)

As $f(x) \in \mathbb{Z}[x]$. Now if $f(x) \in \mathbb{Q}[x]$ and is reducible in $\mathbb{Q}$. But all reducible element in $\mathbb{Q}$ is not necessarility reducible in $\mathbb{Z}$. Hence f(x) may not reducible in $\mathbb{Z}$.

19. Let $I = (x^2 + x + 1)$ in $\mathbb{Z}_n[x]$, $1 \le n \le 10$ Then, $\mathbb{Z}_n[x]/I$ is a field if

(a) $n \le 5$ (b) $n = 2$ (c) $n = 3$ (d) $n = 7$

Solution : (**d**) − Consider I = ($x^2 + x + 1$)

When n = 2 we have $\mathbb{Z}_2 = \{ 0 , 1\}$, But both 0 and 1 are not solution of I. Hence not reducible for n = 2.

When n = 3 , $\mathbb{Z}_2 = \{ 0 , 1 , 2\}$, But both 0 1 and 2 are not solution of I. Hence not reducible for n = 3.

When n = 7 , $\mathbb{Z}_2 = \{ 0 , 1 , 2 , 3 , 4 , 5, 6\}$,

Also , $(2)^2 + 2 + 1 = 4 + 2 + 1 = 7 \equiv 0 \pmod 7$

20. The polynomial x is irreducible in $\mathbb{Z}_n[x]$

(a) for each n (b) for $n \ge 3$ (c) iff n is prime (d) never.

Solution : − Let us consider , f(x) = x

Now for each value of n we can see f(x) is not having any non zero solutions in $\mathbb{Z}_n[x]$.

Option (a)

21. The number of roots of the polynomial $x^{25} - 1$ in $\mathbb{Z}_{37}[x]$ is

(a) 25 (b) 5 (c) 24 (d) 1

Solution : −

Here , $\mathbb{Z}_{37} = \{ 0 , 1 , 2 , 3 , 4 , ------ , 36 \}$

1 is the only solution of x^{25} - 1 = 0 in Z_{37}.

Option (d)

22. Let $f(x) = x^3 - x^2 + 1$

(a) $(f(x))$ is a maximal ideal in $\mathbb{Z}_2[x], \mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$.

(b) $(f(x))$ is a maximal ideal in $\mathbb{Z}_3[x]$ and $\mathbb{Z}_5[x]$ but not in $\mathbb{Z}_2[x]$.

(c) $(f(x))$ is a maximal ideal in $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$ but not in $\mathbb{Z}_5[x]$.

(d) None of the above

Solution : −

f(x) = $x^3 - x^2 + 1$ in $\mathbb{Z}_2[x]$ when x = 1 we get

1 - 1 + 1 = 1 $\not\equiv 0 \ (mod\ 2)$

Hence , f(x) is irreducible in $\mathbb{Z}_2[x]$

For $\mathbb{Z}_3[x]$ we get $\mathbb{Z}_3 = \{ 0 , 1, 2\}$, Here

1 - 1 + 1 = 1 $\not\equiv 0 \ (mod\ 3)$

Again , $2^3 - 2^2 + 1 = 8 - 4 + 1 = 5 \not\equiv 0 \ (mod\ 3)$

Hence , f(x) is irreducible in $\mathbb{Z}_3[x]$

For $\mathbb{Z}_5[x]$ we get $\mathbb{Z}_5 = \{ 0, 1, 2, 3, 4\}$, Here

1 - 1 + 1 = 1 $\not\equiv 0 \ (mod\ 5)$

$2^3 - 2^2 + 1 = 8 - 4 + 1 = 5 \equiv 0 \ (mod\ 5)$

Hence , f(x) is reducible in $\mathbb{Z}_5[x]$

Therefore , f(x) is maximal in $\mathbb{Z}_2[x]$ and $\mathbb{Z}_3[x]$ but not maximal in $\mathbb{Z}_5[x]$.

Option (c)

23. Let $f(x) = x^{10} + x^9 + \cdots + x + 1, g(x) = x^{11} + x^{10} + x^9 + \cdots + x + 1$. Then

(a) $f(x), g(x)$ are both irreducible over $\mathbb{Z}[x]$.

(b) $f(x), g(x)$ are both reducible over $\mathbb{Z}[x]$.

(c) $f(x)$ is irreducible over $\mathbb{Z}[x]$, $g(x)$ is not.

(d) $g(x)$ is irreducible over $\mathbb{Z}[x]$, $f(x)$ is not.

Solution : −

f(x) = $x^{10} + x^9 + ---- + x + 1$

$= (x^{10} + x^9) + (x^8 + x^7) + (x^6 + x^5) + (x^4 + x^3) + (x^2 + x + 1)$

So f(x) is not reducible but if we consider

g(x) = $x^{11} + x^{10} + x^9 + ---- + x + 1$

$= (x^{11} + x^{10}) + (x^9 + x^8) + (x^7 + x^6) + (x^5 + x^4) + (x^3 + x^2) + (x + 1)$

$= x^{10}(x + 1) + x^8(x + 1) + x^6(x + 1) + x^4(x + 1) + x^2(x + 1) + 1(x + 1)$

$= (x^{10} + x^8 + x^6 + x^4 + x^2 + 1)(x + 1)$

But we can see that g(x) is irreducible in $\mathbb{Z}[x]$.

24. The polynomial $f(x) = x$ is

(a) irreducible over any ring R. (b) irreducible but not prime over any ring R.

(c) can be factored in some polynomial ring. (d) has no roots.

Solution : − (a)

We know f(x) = x is not reducible in any $\mathbb{Z}_n[x]$ for any value for n .
Similarly it has not root in any ring R so irreducible over R.

25. In $\mathbb{R}[x]$, Let $I = \{f(x) \in \mathbb{R}[x] : f(2) = f'(2) = f''(2) = 0\}$ and $J = \{f(x) \in \mathbb{R}[x] : f(2) = 0, f'(3) = 0\}$
(a) I, J are ideals in $\mathbb{R}[x]$. (b) I is an ideal, J is not.
(c) Neither I nor J is an ideal. (d) I is a prime ideal in $\mathbb{R}[x]$.

Solution :- Option (b)

I = { f(x) ∈ $R[x]$ where f(2) = f'(2) = f''(2) = 0 }
Let f(x) = $a_0 + a_1x + a_2x^2 + ----- -$ for $a_i \in R$
$f(2) = 0$
$\Rightarrow a_0 + 2a_1 + 4a_2 + 8a_3 + \ ---- \ = 0$---------------(1)
$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$
$\Rightarrow f'(2) = 0$
$\Rightarrow a_1 + 2a_2 \times 2 + 3a_3 \times 4 + 4a_4 \times 8 = 0$
$\Rightarrow a_1 + 4a_2 + 12\ a_3 + 32a_4 = 0$-------------------------(2)
f''(x) = $2a_2 + 6a_3x + 12a_4x^2 + \ --------- -$
$f''(x) = 0$
$\Rightarrow 2a_2 + 12a_3 + 12a_4 \times 4 + \ ----- - \ = 0$
$\Rightarrow \ 2a_2 + 12a_3 + 48a_4 + \ ---- - = 0$
From (1)
$a_0 = -2(a_1 + 2a_2 + 4a_3 + 8a_4 + \ ----- -)$
$a_1 = -4\ (\ a_2 + 6a_3 + 16a_4 + \ ------ -)$
$2a_2 = -12a_3 - 48a_4 \ --------- -$
$\Rightarrow a_2 = -6a_3 - 24a_4 \ ------- -$
= -6 ($a_3 + 4a_4 + \ ------ -$)
$\Rightarrow \ a_1 = -4\ (-6a_3 - 24a_4 + 6a_3 + 16a_4)$
= -4 (-8 a_4)
$= 32\ a_4$
$\Rightarrow \ a_0 = -2\left\{32a_4 - 12a_3 - 48a_4 + 4a_3 + 8a_4 \ + \ ----- -\right\}$
= - 2[-8 $a_3 - 8a_4$]
$= 16\ (\ a_3 - a_4)$

So , I is an principal ideal of $R[x]$

Similarly we can check that J is not an ideal