PAPER I: Probability and Distribution Theory

UNIT I: Probability

After studying this unit you will be able to:

- Develop an understanding of the theory of probability and rules of probability.
- Apply probability rules and concepts within a practical and business context.
- Demonstrate knowledge of the importance of probability in practical situation.

The mathematical, statistical, or empirical, attitude toward probability has been developed by R. A. FIsher and R. von MIses. The notion of sample space comes from Von Mises. The theory is limited to one particular aspect of "chance." The intuitive notion of probability is connected with inductive reasoning and with judgments such as "Paul is probably a happy man," "Probably this book will be a failure," "Fermat's conjecture is probably false."

Suppose that we have such an experiment, but the experiment is of such a nature that a collection of every possible outcome can be described prior to its performance. If this kind of experiment can be repeated under the same conditions, it is called a random experiment, and the collection of every possible outcome is called the experimental space or the sample space. We denote the sample space by S.

Example 1. In the toss of a coin, let the outcome tails be denoted by T and let the outcome heads be denoted by H. If we assume that the coin may be repeatedly tossed under the same conditions, then the toss of this coin is an example of a random experiment in which the outcome is one of the two symbols T or H; that is, the sample space is the collection of these two symbols. For this example, then, $C = \{H,T\}$.

Example 2. In the cast of one red die and one white die, let the outcome be the ordered pair (number of spots up on the red die, number of spots up on the white die). If we assume that these two dice may be repeatedly cast under the same conditions, then the cast of this pair of dice is a random experiment. The sample space consists of the 36 ordered pairs: $C = \{(1,1),\ldots,(1,6),(2,1),\ldots,(2,6),\ldots,(6,6)\}$

Before starting probability we will just go through definitions of permutation and combination.

Permutations and Combinations

Permutations and Combinations are **Mathematical terms**. Permutation is the **arrangement of objects in which order is priority.** Combination is the arrangement of objects in which order is irrelevant. The fundamental **difference between permutation** and **combination** is the order of objects, in **permutation** the order of objects is very important, i.e. the arrangement must be in the stipulated order of the number of objects, taken only some or all at a time. The notation for permutation is P (n, r) or ^nP_r which is the number of permutations of n things if only r are selected.

$$^nP_r = \frac{n!}{(n-r)!}$$

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Problems:

1. 6 cards are to be send to 4 persons, in how many ways this can be done?

2. In how many ways 3 pencils can be selected from 5 pencils?

3. From a group of 7 boys and 6 girls, 3 boys and 4 girls is to be selected. In how many ways this can be done?

Introduction to Probability

Probability means possibility or chance. We are certain about “rising of the sun every day”, about “there are 7 days in a week” etc. However there are many things where we are not sure about the occurrence or the outcome of the incident, in those cases we use the words probably or likely or possibly.

For example, “Probably it will rain to night”, “it is quite likely that there will be a good yield of crop this year” and so on. But the terms probably, quite likely are all relative terms of uncertainty. Probability is a numerical measure of uncertainty – a number that conveys the strength of out belief in the occurrence of an uncertain event.

The theory of probability was largely developed by European mathematicians such as Galileo, Pascal and others.

To find a measure for probability it is necessary to have the concept of few terms which we discussed below.

Random Experiment or Trial

An operation or experiment conducted under identical conditions which has a number of possible outcomes is called Random Experiment.

Example :

1. Tossing a coin
2. Throwing a dice
3. Selecting a card form a pack of cards

Sample Space

The set of all possible outcomes of a random experiment is called sample space. The elements of the sample space are called sample points. Sample space is denoted by S.

Example :

1. In an experiment of
   throwing a coin S={H,T]
2. In an experiment of
   throwing a dice
   S={1,2,3,4,5,6}

The number of sample points in a sample space of random experiment is denoted by *n* (*s*). For example (1) *n* (S) = 2, and example (2) *n* (S) = 6

Discrete Sample Space

A sample space containing finite or countably infinite number of points is called a discrete sample space. **Example:** If the random experiment is throwing a coin, sample space = S = {1,2,3,4,5,6}

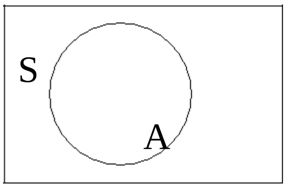
Continuous Sample Space

A sample space containing uncountable sample points is called a continuous sample space. Example: All rational numbers between 5 and 10

Event

Any subset of the sample space S is called an event. If S is a sample

space and A is a subset of S (i.e., A⊂ S), then A is called an event.



Using Venn diagram, we get,

Example :

In an experiment of throwing dice where S = {1, 2, 3, 4, 5, 6}, the event of getting even numbers is A = {2, 4, 6}

Clearly A⊂ S

The number of sample points in A is denoted by *n* (A). For the above experiment, *n* (A) = 3

Types of Events

1. Certain Event

If sample points in an event are same as sample points in sample space of that random experiment, then the event is called a certain event.

Example: Getting any number between1 to 6 on a dice is a certain event.

2. Impossible Events

An event which never occurs or which has no favourable outcomes is called an impossible event. In other words, the event corresponding to the set φ (null set) is called an impossible event.

Example: Getting a number 7 on a dice is an impossible event.

3. Mutually Exclusive Events

Events are said to mutually exclusive if the happening of any of them restricts the happening of the others i.e., if no two or more of them can happen together or simultaneously in the same trial.

Example : In tossing a coin event head and tail are mutually exclusive.

Note: If A & B are mutually exclusive events of sample space S, then $A \cap B = \varphi$.

4. Equally Likely Events

Events are said to be equally likely if they have equal choice to occur. In other words, outcomes of a trial are said to be equally likely if taking into consideration all relevant evidences, there is no reason to prefer one with respect to other.

Example: In throwing a dice all the six faces are equally likely to occur.

5. Exhaustive Events

If the sample points of the events taken together constitute the sample space of the random experiment, the events are called exhaustive events.

Note: If A & B are exhaustive events of sample space S, then
A U B = S.

Example: Random Experiment: Throwing a dice

S = {1,2,3,4,5,6}

A = Event of odd numbers = {1, 3, 5}

B = Event of even numbers = {2, 4, 6}

C = Event of multiple of 3 = {3, 6}

Then, A UB = {1, 2, 5, 3,4,6} = S

A U C = {1, 3,5,6} ≠ S

Here A and B are called exhaustive events.

Here A and C are not exhaustive events.

6. Complementary Event (A’, A^c, $\bar{A}$)

If A is an event in sample space S, then the non-occurrence event of A is called Complementary event of A. Two events A and B are called complementary events, if A and B exhaustive as well as mutually exclusive events. In other words, A and B are called complementary events if A U B = S and A ∩ B = φ.

7. Equivalent events: If $A \subset B$ Band $B \subset A$, then A and B are equivalent events and signify that every point of A is contained in B; they are read, respectively, "A implies B" and "B is implied by A".

Examples. (a) If A and B are mutually exclusive, then the occurrence of A implies the non-occurrence of B and vice versa. Thus $A \cap B = 0$ means the same as $A \subset$ B' and as $B \subset$ A'.

(b) The event A -AB means the occurrence of A bat not of both A and B. Thus A - AB = AB'.

Example :

Random Experiment : Throwing a dice

S = {1, 2, 3, 4, 5, 6} A = {1, 2}

B = {3. 4. 5 6}

As A U B = S and A ∩ B = φ, A and B are complementary events.

Complementary event of A is denoted by A^c, $A^{/}$ or A.

MATHEMATICAL OR CLASSICAL DEFINITION OF PROBABILITY OF AN EVENT

If the sample space S of a random experiment consists of n equally likely, exhaustive and mutually exclusive sample points and *m* of them are favourable to an event A, then the probability of event A is given by

$$\text{P(A)} = \frac{m}{n} \quad = \frac{\text{Number of Sample Point in A}}{\text{Number of Sample Point in S}} = \frac{n(A)}{n(S)}$$

Note: $0 \le m \le n$

$$\frac{0}{n} \le \frac{m}{n} \le \frac{n}{n} \Rightarrow 0 \le P(A) \le 1$$

Example : 1

Two unbiased dice are thrown. Find the probability that:

i) Both the dice show same number.

ii) First die shows 6.

iii) The total of the numbers on the dice is 8.

Solution:

In a random throw of two dice, the total number of cases is given below :

$$\begin{array}{llllllll}
S & = & \{(1, 1), & (2, 1), & (3, 1), & (4, 1), & (5, 1), & (6, 1), \\
 & & (1, 2), & (2, 2), & (3, 2), & (4, 2), & (5, 2), & (6, 2), \\
 & & (1, 3), & (2, 3), & (3, 3), & (4, 3), & (5, 3), & (6, 3), \\
 & & (1, 4), & (2, 4), & (3, 4), & (4, 4), & (5, 4), & (6, 4), \\
 & & (1, 5), & (2, 5), & (3, 5), & (4, 5), & (5, 5) & (6, 5), \\
 & & (1, 6), & (2, 6), & (3, 6), & (4, 6), & (5, 6) & (6,6)\}
\end{array}$$

Here, $n\,(S) = 36$

i) A : Both the dice show same number

$$= \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$n\,(A) = 6$$

$$\text{P(A)} = \frac{n(A)}{n(S)} = \frac{6}{36} \quad = 1/6$$

ii) B : First die show 6

$$= \{(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

$$n(B) = 6$$

$$P(A) = \frac{n(B)}{n(S)} = \frac{6}{36} = 1/6$$

iii) C : Total of the number on the dice is 8

$$= \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\} \Rightarrow n(C) = 5$$

$$= P(C) = \frac{n(C}{n(S} = 5/36$$

Example : 2

Two unbiased coins are tossed simultaneously. Find the probability of getting –

i) at least one tail

ii) majority of heads

Solution :

Let S be the sample space

$$S = \{(H, H), (H, T), (T, H), (T, T)\}$$

$$n(S) = 4$$

i) A : At least one tail

$$= \{(H, T), (T, H), (T, T)\}$$

$$n(A) = 3$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{3}{4}$$

ii) B : Majority of heads

$= \{(H, H)\}$

$n(B) = 1$

$$P(B) = \frac{n(B)}{n(S)} = \frac{1}{4}$$

Example :3

A box contains 20 tickets numbered from 1 to 20. A ticket is drawn randomly from the box. Find the probability that the number on the ticket is

i) Divisible by 5

ii) Not divisible by 2

iii) Divisible by 3 and 4.

iv) Divisible by 3 or 4.

Solution :

Let S be the sample Space.

$S = \{1, 2, 3, \ldots\ldots\ldots\ldots, 20\}$

$n(S) = 20$

i) A : Divisible by 5

A {5, 10, 15, 20}

$n(A) = 4$

$$P(A) = \frac{n(A)}{n(S)} = \frac{4}{20} = \frac{1}{5}$$

ii) B : Not divisible by 2

$B = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19\}$

$n(B) = 10$

$$P(B) = \frac{n(B)}{n(S)} = \frac{10}{20} = \frac{1}{2}$$

iii) C : Divisible by 3 and 4.

C = {12}

n (C) = 1

$$P(C) = \frac{n(C)}{n(S)} = \frac{1}{20}$$

iv) D = Divisible by 3 or 4.

D = {3, 4, 6, 8, 9, 12, 15, 16, 18, 20}

n (D) = 10

$$P(D) = \frac{n(D)}{n(S)} = \frac{10}{20} = \frac{1}{2}$$

Example: 4

A bag contains 10 while and 11 black balls. If two balls are drawn simultaneously from the bag. Find the probability of getting (i) both white balls, (ii) one white and one black ball, (iii) no white ball.

Solution:

Let S be the sample space.

The bag contains 10 white + 11 black = 21 balls

n (S) = Total number of cases = $^{21}C_2$

$$= \frac{21!}{2! \times 19!} = \frac{21 \times 20}{2} = 210$$

(i) A = Both white balls

n (A) = Favourable number of cases = $^{10}C_2$

$$= \frac{10!}{2! \times 8!} = \frac{10 \times 9}{2} = 45$$

$$P(A) = \frac{n(A)}{n(S)} = \frac{45}{210} = 0.2143$$

(ii) n (B) =Favourable number of cases =

$^{10}C_1 \times {}^{11}C_1 = = 10 \times 11 = 110$

$$P(B) = \frac{n(B)}{n(S)} = \frac{110}{210} = 0.5238$$

(iii) n (C) =Favourable number of cases

$$P(B) = \frac{n(C)}{n(S)} = -$$

Example 5: There are 50 tickets with numbers 1 to 50. 5 tickets are chosen at random and arranged in increasing order. Compute the probability that the middle number is 30.

Solution: n(S) = Total number of cases = $^{50}C_5$

The 5 tickets are arranged in increasing order, i.e. x1<x2<x3<x4<x5.

The task is to find P(x3 = 30).

Here 1, 2, 3,……….,28, 29, 30, 31, 32, ………..49, 50.

First two numbers are selected from 29 in ${}^{29}C_2$ and last two numbers are selected from 31 to 50, i.e., from 29 numbers in ${}^{20}C_2$ ways.

So P(x3 = 30) = ${}^{29}C_2 \times {}^{20}C_2 / {}^{50}C_5 = 0.0364$

Example 6: If m biscuits are to be distributed to n children randomly, what is the probability that a particular child receives k biscuits?

Solution:

n (S) = Total number of cases = n^m

n (A) = Number of cases where a particular student will get k biscuits.

k biscuits can be selected from m biscuits by ${}^{m}C_k$ ways.

Now we have m-k biscuits to distribute among n-1 children.

It can be distributed in $(n-1)^{m-k}$ ways.

P(A) = $\frac{n(A)}{n(S)} = \frac{{}^{m}C_k \times (n-1)^{m-k}}{n^m}$

Example 7: The face cards are removed from a pack of cards. Out of the remaining 4 cards are selected. What is the probability that they belong to different suits?

Solution:

As there are 12 face cards, after removing them there are 40 cards remaining. From 40 cards, 4 cards of different suits are selected. Let A be the event that the cards belong to different suit.

In 40 cards in each suit there are 10 cards.

n (S) = ${}^{40}C_4$

n (A) = $10 \times 10 \times 10 \times 10 = 10^4$

P(A) = $\frac{n(A)}{n(S)} = \frac{10^4}{{}^{40}C_4}$

Limitations of Mathematical Probability:

1. If the various outcomes of the trial are not equally likely.
2. If the exhaustive number of outcomes in a trial is infinite.
3. If the experiment cant’ be repeated under identical conditions

Statistical or Empirical Probability

If a trial is repeated a number of times under essentially homogenous and identical conditions, then the limiting value of the ratio of the number of times the event happens to the number of trials becomes indefinitely large, is called the probability of the event.

If in n trials, an event E happens m times, then the probability of E is P(E) = $\lim_{n \to \infty} \frac{m}{n}$

Some Important Results of Set Theory

Set theory is a branch of mathematical logic that studies sets, which informally are collections of objects or things of similar type. Although any type of object can be collected into a set, set theory is applied most often to objects that are relevant to mathematics. Sets are usually denoted by A, B, C. The followings are some examples of sets.

A = The set of integers = {1, 2, 3, 4 …},

B = The set of Vowels = {a, e, i, o, u}

C = The set of days in the week

= {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

The objects in the set are called elements or members of the set.

$x \in A$ ⇒ x is an element of the set A

$x \notin A$ ⇒ x is not an element of the set A

Equality of Sets

Two sets are equal it and only if they have the same elements.

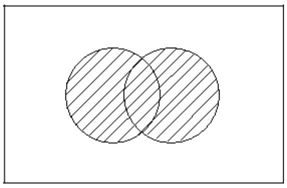
Subsets

A is a subset of B if and only if every elements of A is an element of B, we write it as $A \subset B$, we can also say as " B includes A".

B

Union

The union of the set A and the set B is the set that contains all the elements that belong to A or to B, written AU B.

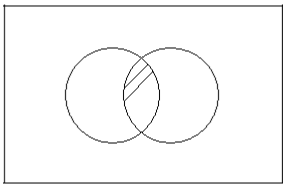


The shaded portion is A U B.

Intersection

The intersection of the set A and the set B is the set that contains all the elements that belong to A and B both, written as A ∩ B.

The shaded part is A ∩ B.



Complementary set

The element of universal set S which do not belong to the subset A, forms a set which is called complement of A and is denoted by A^c or A' or A .

Universal and Empty Set

In a set theory, a universal set is a set which contains all objects, including itself.

The complement of universal set in called empty set, or null set.

Universal set is denoted by S and empty set is denoted by φ.

Example: Suppose we have a spinner with the numbers 1 through 10 on it. The experiment is to spin the spinner and record the number spun. Then S ={1,2,...,10}. Define the events A, B, and C by A = {1,2}, B ={2,3,4}, and C = {3,4,5,6}, respectively. A^c = {3,4,...,10}; A∪B ={1,2,3,4}; A∩B = {2} A∩C = φ; B∩C = {3,4}; B ∩C ⊂ B; B∩C ⊂ C

A∪(B ∩C)={1,2}∪{3,4}= {1,2,3,4} (A∪B)∩(A∪C)={1,2,3,4}∩{1,2,3,4,5,6}= {1,2,3,4}

Note: A∩ (B ∪C)=(A∩B)∪(A∩C)

A∪ (B ∩C)=(A∪B)∩(A∪C)

The next two identities are collectively known as **De Morgan's Laws.** For any sets A and B,

$(A\cap B)^c = A^c \cup B^c$ and $(A\cup B)^c = A^c \cap B^c$.

Axiomatic Definition of Probability: Let S be a sample space and let B be the set of events. Let P be a real-valued function defined on B and the range is [0,1]. Then P is a probability set function if P satisfies the following three conditions:

1. P (A) ≥ 0, for all A ∈ B, 2. P(S) = 1

3. If {An} is a sequence of events in B and Am ∩ An = φ for all m ≠ n,

Then, $P\left(\coprod_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$

A collection of events whose members are pairwise disjoint, as in (3), is said to be a mutually exclusive collection and its union is often referred to as a disjoint union. The collection is further said to be exhaustive if the union of its events is the sample space, in which case $\sum_{n=1}^{\infty} P\ (A_n)$ =1 We often say that a mutually exclusive and exhaustive collection of events forms **a partition of S.**

Theorem 1: For each event $A \in S$, $P(A)=1-P(A^c)$.

Proof: We have $S = A \cup A^c$ and $A \cap A^c = \varphi$.

Thus, from (2) and (3) of Definition, it follows that

$P(A \cup A^c) = P(A) + P(A^c) = P(S) = 1$ which implies $P(A)=1-P(A^c)$.

Theorem 2. The probability of the null set is zero; that is, $P(\varphi)=0$.

Proof: In Theorem 1, take $A = \varphi$ so that $A^c = S$. Accordingly, we have $P(\varphi)=1-P(S)=1-1=0$ and the theorem is proved.

Theorem 3. If A and B are events such that $A \subset B$, then

$P(A) \leq P(B)$.

Proof: Now $A \subset B \Rightarrow n(A) \leq n(B)$

$$\Rightarrow \frac{n\ (A)}{n\ (S)} \leq \frac{n\ (B)}{n\ (S)} \Rightarrow P(A) \leq P(B)$$

Theorem 4. For each $A \in B$, $0 \leq P(A) \leq 1$.

Proof: Since $\varphi \subset A \subset S$, we have by Theorem 3 that $P(\varphi) \leq P(A) \leq P(S)$ or $0 \leq P(A) \leq 1$

Theorem 5. (Addition Theorem)

If A and B are events in S, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof: Each of the sets $A \cup B$ and B can be represented, respectively, as a union of nonintersecting sets as follows: $A \cup B = A \cup (A^c \cap B)$ and

$B = (A \cap B) \cup (A^c \cap B)$.

That these identities hold for all sets A and B follows from set theory.

Thus, from (3) of Definition, $P(A \cup B) = P(A) + P(A^c \cap B)$ and

$P(B) = P(A \cap B) + P(A^c \cap B)$.

If the second of these equations is solved for $P(A^c \cap B)$ and this result is substituted in the first equation, we obtain

P(A∪B)=P(A) +P(B)−P(A∩B). This completes the proof.

Theorem 6.

For n =3, the above formula is:

P(A1UA2UA3)=

P[A1]+P[A2]+P[A3]−P[A1∩A2]−P[A1∩A3]−P[A2∩A3] +

P[A1∩A2∩A3].

Proof: Refer class notes

For n =4, the above formula is:

P(A1UA2UA3 U A4)= P[A1]+P[A2]+P[A3] + P(A4)−P[A1∩A2]−P[A1∩A3] P(A1∩A4)−P[A2∩A3]-……..

+ P[A1∩A2∩A3] + P(A1∩A2∩A4) +…….- P(A1∩A2∩A3∩A4)

More generally, for n events

A1,A2,...An ∈S we have:

$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}\sum P(A_i A_j) + \sum_{i<j<k}\sum\sum P(A_i A_j A_k) - \ldots\ldots$

$+ (-1)^{n+1} P(A_1 A_2 \ldots\ldots A_n)$

Thus for arbitrary events A, B, ... the inequality

P (A∪B) ≤ P (A) + P (B) holds.

In the special case where the events A1, A2, . .. are mutually exclusive, we have

P (A1UA2U…..U An) ≤ $P(A1) + P(A2) + \cdots + P(An)$

This is referred to as **Boole's inequality.**

Conditional Probability

The conditional probability of an event A is the probability that the event will occur given the knowledge that an event B has already occurred. We say probability of the event A given the event B has already occurred and denote it by P (A / B).

If the events A and B are such that the occurrence of A doesn't depend upon occurrence of event B, (A and B are independent event), the conditional probability of event A given event B is simply the probability of event A, that is P (A).

Similarly, probability of event B given that event A has already occurred is denoted by P (B / A).

Then we define the conditional probability of A given B to be $P(A/B) = P(A \cap B)/ P(B)$.

Then we define the conditional probability of B given A to be $P(B/A) = P(A \cap B)/ P(A)$.

Note: 1. $P (B|A) \geq 0$.

2. $P(A|A) = 1$.

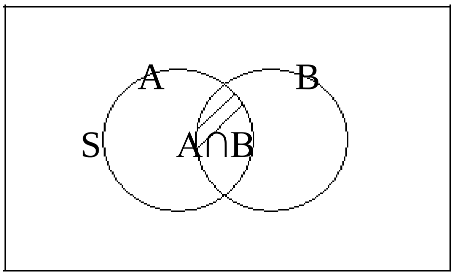
3. $P \left(\coprod_{n=1}^{\infty}(Bn|A) \right) = \sum_{n=1}^{\infty} P$ (B_n / A, provided that B1,B2,... are mutually ex-clusive events.

10.2.3 Multiplication Theorem

If A and B are two events of a sample space S associated with an experiment, then the probability of simultaneous occurrence of events A and B is given by

$$P (A \cap B) = P(A) P(B/A) = P(B) P(A/B)$$

Where P(B/A) is the conditional probability of B given A has already occurred and P(A/B) vice versa.



Independent Events

Two events A and B are independent of each other if the occurrence or non-occurrence of one does not affect the occurrence of the other.

i.e., $P(A/B) = P(A)$

$P(B/A) = P(B)$

Then $P(A\cap B) = P(A)\,P(B)$

In general if there are three independent events A, B and C associated with an experiment, then

$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$.

Note: 1. If A and B are disjoint, then by countable additivity of P, we have $P(A\cup B) = P(A)+ P(B)$.

2. If events A1,...,An are independent

$P(\cap A_i) = \prod P((A_i)$.

3. P (atleast one event) = P (no event)

$\Rightarrow P(A \cup B) = 1 - P(A^c \cap B^c)$

Theorem of Total Probability

Let A1,...,Ak be a (finite) partition of S, and let B ⊆ S.

P(B) = $\sum_{i=1}^{k} P(B/\text{Ai})$ P(Ai)

Proof:

A1,...,Ak form a partition of S, and B ⊆ S,

So B = (B ∩A1)∪......∪(B ∩Ak)

⇒ P(B) = P (B ∩A1)∪......∪(B ∩Ak)

⇒ P(B) = $\sum_{i=1}^{k} P\ (B \cap Ai)$ since (B∩A1),…,(B∩Ak) are disjoint or mutually exclusive events.

⇒ P(B) = $\sum_{i=1}^{k} P$ (B | Ai) P (Ai) (From the definition of conditional probability)

Extension: The theorem still holds if A1,A2,... is a (countably) infinite a partition of S, and B ⊆ S, so that

P(B) = $\sum_{i=1}^{\infty} P$ (B | Ai) P (Ai)

Problems:

Example : 1

Find the probability that a card drawn from a pack of cards will be a red or a picture card.

Solution :

Let A = Event of getting red card

B = Event of getting picture card

P(A) = $\frac{26}{52} = \frac{1}{2}$ P(B) = $\frac{12}{52} = \frac{3}{13}$

There are 6 red cards which are picture cards,

$P(A \cap B) = \frac{6}{52}$

P (The card is red or picture) = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\frac{1}{2} + \frac{1}{4} - 6/52 = 8/13$

Example: 2

Two balls are drawn from a bag one by one with 2 white and 3 black balls. What is the probability that the second ball is white ?

Solution :

Let A1 be the event that the first ball is white.

Let B be the event that the second ball is white.

Let A2 be the event that the first ball is black.

Then from the formula, $P(B) = \sum_{i=1}^{k} P(B \mid Ai)\, P(Ai)$

We get, $P(B) = P(B \mid A1)\, P(A1) + P(B \mid A2)\, P(A2)$

$$= \frac{1}{4} \times \frac{2}{5} + \frac{2}{4} \times \frac{3}{5} = \frac{2}{5}$$

Example: 3

Suppose we draw two cards from a shuffled set of 52 playing cards. What is the probability that the second card is a Queen?

Solution:

P(2nd card Q)= P(2nd card Q|1st card Q) · P(1st card Q) + P(2nd card Q|1st card not Q) · P(1st card not Q)

$$= \frac{3}{51} \times \frac{4}{52} + \frac{4}{51} \times \frac{48}{52} = \frac{4}{52}$$

Example: 4

A and B are two independent events such that, P (A) = 0.2 and P (B) = 0.4. Find the probability that (i) both A and B will occur (ii) only A occurs, (iii) only B will occur, (iv) at least one will occur, (v) none will occur.

Solution:

$P(A) = 0.2$ $P(A^c) = 1-0.2=0.8$ $P(B) = 0.4$ $P(B^c) = 1-0.4=0.6$

(i) P (both A and B will occur) = $P(A \cap B) = P(A)\,P(B)$ [Since A & B are Independent] $= 0.2 \times 0.4 = 0.08$

(ii) P (only A occurs) = $P(A \cap B^C) = P(A)\,P(B^C)$ [Since A & B^C are Independent] $= 0.2 \times 0.6 = 0.12$

(iii) P (only B occurs) = $P(A^C \cap B) = P(A^C)\,P(B)$ [Since A^C & B are Independent] $= 0.8 \times 0.4 = 0.32$

(iv) P (at least one will occur) = $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= 0.2 + 0.4 - 0.08 = 0.52$

(v) P (none will occur) = $P(A^c \cap B^c) = 1 - P(A \cup B)$ $1 - 0.52 = 0.48$

Example: 5

A commerce graduate can get offer from three companies A, B and C. The chances of getting offer from company A is 20%, from B 16%, from C 14% , from A and B both 8%, from A and C both 5%, from B and C both 4% and from all three is 2% . Find what percentage he gets at least one offer.

Solution:

$P(A) = 0.2$ $P(B) = 0.16$ $P(C) = 0.14$ $P(A \cap B) = 0.08$

P (A∩C) = 0.05   P (B∩C) = 0.04   P (A∩B∩C) = 0.02

P (he gets at least one offer)   = P (AUBUC)

= P (A) + P(B) + P(C) – P(A∩B) – P(B∩C) – P(A∩C) + P(A∩B∩C)

= 0.2 + 0.16 + 0.14 – 0.08 – 0.05 – 0.04 + 0.02 = 0.35

Example: 6

The odds in favour of A hitting a target are 3 : 4 and odds against B hitting a target are 1 : 2 If both of them shoot the target independently, what is the probability of (i) both hit the target, (ii) only A hits the target (iii) at least one of them hits the target. (iv) none hits the target.

Solution:

P(A) = $\frac{3}{7}$ P(A^c) = $\frac{4}{7}$ P(B) = $\frac{2}{3}$ P(B^c) = $\frac{1}{3}$

(i) P (both A and B hit the target) = P(A ∩ B) = P(A) P(B) [Since A & B are Independent

$= \frac{3}{7} \times \frac{2}{3} = \frac{6}{21}$

(ii) P (only A hits the target) = P(A ∩ B^C) = P(A) P(B^C) [Since A & B^C are Independent]

$= \frac{3}{7} \times \frac{1}{3} = \frac{3}{21}$

(iv) P (at least one will hit) = P (A U B) = P (A) + P (B) – P(A∩B) $= \frac{3}{7} + \frac{2}{3} - \frac{6}{21} =$

$\frac{9+14-6}{21} = \frac{17}{21}$

(v) P (none will occur) = P (A^c ∩ B^c) = 1 - P (A U B) = $1 - \frac{17}{21} = \frac{4}{21}$

Example 7:

Two dice are tossed. Let A be the event that the total is an odd number, B be the event that ace is on first dice and C be the event of total 7. Discuss the independence of the events A, B and C.

Baye's Theorem:

Baye's Theorem is a direct application of conditional probability. In probability theory and statistics, Bayes' theorem describes the probability of an event, based on prior knowledge of conditions that might be related to the event. For example, if the risk of developing health problems is known to increase with age, Baye's theorem allows the risk to an individual of a known age to be assessed more accurately than simply assuming that the individual is typical of the population as a whole.

The probability P (A / B) of "A assuming B is given" is given by the formula

$P\ (A/B) = \frac{\mathrm{P}\ (\mathrm{A}\cap\mathrm{B})}{P\ (B)}$.

Similarly the probability P (B / A) of "B assuming A is given" is given by the formula

$P\ (B/A) = \frac{\mathrm{P}\ (\mathrm{A}\cap\mathrm{B})}{P\ (A)} \Rightarrow P\ (A\cap B) = P(B/A)\ P(A)$

Combining above two formulas we can write

P(A | B) = P(B | A)P(A) / P(B)

Let Let A1,...,Ak be a (finite) partition of S, and let B ⊆ S.

Then, $P(B) = \sum_{i=1}^{k} P(B/\mathrm{Ai})\ \mathrm{P(Ai)}$

$$P(Ai/B) = P(B\ /Ai)\ \ P(Ai)\ /\ P(B)$$

$$\Rightarrow P(A_i\ /B) = \frac{P(B\ I\ Ai)P(Ai)}{\sum_{i=1}^{k} P\ (\mathrm{B} \mid \mathrm{Ai})\ \mathrm{P}\ (\mathrm{Ai})}$$

Example

You might wish to find a person's probability of having rheumatoid arthritis if they have hay fever. In this example, "having hay fever" is the test for rheumatoid arthritis (the event).

A would be the event "patient has rheumatoid arthritis." Data indicates 10 percent of patients in a clinic have this type of arthritis. P(A) = 0.10

B is the test "patient has hay fever." Data indicates 5 percent of patients in a clinic have hay fever. P(B) = 0.05

The clinic's records also show that of the patients with rheumatoid arthritis, 7 percent have hay fever. In other words, the probability that a patient has hay fever, given they have rheumatoid arthritis, is 7 percent. P(B | A) =0.07

Substituting these values into the theorem:

P(A | B) = (0.07 × 0.10) / (0.05) = 0.14

So, if a patient has hay fever, their chance of having rheumatoid arthritis is 14 percent. It's unlikely a random patient with hay fever has rheumatoid arthritis.

More generally for a finite number of mutually exclusive and exhaustive events H_i (I = 1, 2, …… n), i.e., events that satisfy, $H_i \cap H_j = \Phi$ for all $i \neq j$ and H_1 H_2 …. H_n = S (Sample Space),

Baye’s Theorem states that, $P\ (H_i\ /\ A) = \frac{\mathrm{P}\ (\mathrm{A}\ /\ \mathrm{H}i)\ \mathrm{P}\ (\mathrm{H}i)}{\sum_{i=1}^{n} \mathrm{P}\ (\mathrm{A}\ /\ \mathrm{H}i)\ \mathrm{P}\ (\mathrm{H}i)}$

Example : 1

Suppose there are two bags with first bag contains 3 white and 2 black balls, second bag contains 2 white and 4 black balls. One ball is transferred from first bag to second bag and then a ball is drawn from the later and it is found to be white. What is the probability that the transferred ball is white?

Solution:

Let B be the event of drawing a white ball from the second bag. A_1 is the event of transferring a white ball from bag 1 and A_2 is the event of transferring a black ball from bag 1.

$P(A1) = 3/5, \quad P(A2) = 2/5, \quad P(B/A1) = 3/7, \quad P(B/A2) = 2/7$

P (Transferred ball was white given that the ball drawn is white)

$$= P(A_1/B) = \frac{P\,(B/A1)\;P\,(A1)}{P(B/A1)P(A1)+P(B/A2)P(A2)}$$

$$= \frac{(3/7)\times(\frac{3}{5})}{\left(\frac{3}{7}\right)\times\left(\frac{3}{5}\right)+(2/7)\times(\frac{2}{5})}$$

$$= 9/13$$

Example : 2

Three firms A, B, C supply 25%, 35% and 40% of chairs needed to college. Past experience shows that 5%, 4% and 2% of the chairs produced by these companies are defective. If a chair is found to be defective, what is the probability that chair was supplied by firm A.

Solution:

Let D be the event of selecting defective chair. Let A, B and C are the events of chair supplied from firms A, B and C.

P(A) = 0.25, P(B) = 0.35, P(C) = 0.40

P(D/A) = 0.05, P(D/B) = 0.04, P(D/C) = 0.02

P (a chair is found to be defective given it was supplied by firm A.)

$$= \text{P (A/D)} = \frac{\text{P (D/A) P (A)}}{\text{P}\left(\frac{\text{D}}{\text{A}}\right)\text{P(A)}+\text{ P}\left(\frac{\text{D}}{\text{B}}\right)\text{P(B)}+\text{ P(D/C)P(C)}}$$

$$= \frac{0.05 \times 0.25}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4}$$

$$= \frac{00125}{0.0345}$$

= 0.36

Example 3:

Bag I contains 4 white and 6 black balls while another Bag II contains 4 white and 3 black balls. One ball is drawn at random from one of the bags and it is found to be black. Find the probability that it was drawn from Bag I.

Solution:

Let A1 be the event of choosing the bag I, A2 the event of choosing the bag II and B be the event of drawing a black ball.

Then, $P(\text{A1}) = P(A2) = 12$

Also, $P(B|\text{A1})$ = P (drawing a black ball from Bag I) = 6/10

$P(B|A2)$ = P (drawing a black ball from Bag II) = 3/7

By using Bayes’ theorem, the probability of drawing a black ball from bag I out of two bags,

$$P(A1|B) = \; = \frac{\mathrm{P\ (B/A1)\ P\ (A1)}}{\mathrm{P(B/A1)P(A1)+P(B/A2)P(A2)}}$$

$$= \frac{(6/10)\times(\frac{1}{2})}{\left(\frac{6}{10}\right)\times\left(\frac{1}{2}\right)+(3/7)\times(\frac{1}{2})} \quad =$$

Example 4:

A man is known to speak truth 2 out of 3 times. He throws a die and reports that number obtained is a four. Find the probability that the number obtained is actually a four.

Solution:

Let *B* be the event that the man reports that number four is obtained.

Let *A*1 be the event that four is obtained and *A*2 be its complementary event.

Then, $P(A1)$ = Probability that four occurs = 1/6

$P(A2)$ = Probability that four does not occurs = 1 – $P(E1)$ = 1 −1/6 = 5/6

Also, $P(B|\mathrm{A1})$ = Probability that man reports four and it is actually a four = 2/3

$P(B|A2)$ = Probability that man reports four and it is not a four = 1/3

By using Bayes’ theorem, probability that number obtained is actually a four,

$$P(A1|B) = \; = \frac{\mathrm{P\ (B/A1)\ P\ (A1)}}{\mathrm{P(B/A1)P(A1)+P(B/A2)P(A2)}}$$

$$= \frac{(2/3)\times(\frac{1}{6})}{\left(\frac{2}{3}\right)\times\left(\frac{1}{6}\right)+(1/3)\times(\frac{5}{6})} \quad =$$

Probability of Realization of at least one event out of n events:

$\boldsymbol{P_1 = P(\cup_{i=1}^{n} A_i)}$ **= S1 – S2 + S3 - ………….. ± $\boldsymbol{S}$n**

Proof:

Al and A2 are two events, then A = Al U A2 denotes the event that either Al or A2 or both occur.

P(A1$\cup A2) = P(A1) + P(A2) - P(A1 \cap A2)$

For n =3, the above formula is:

P(A1UA2UA3)=

P[A1]+P[A2]+P[A3]−P[A1∩A2]−P[A1∩A3]−P[A2∩A3] +

P[A1∩A2∩A3].

For n =4, the above formula is:

P(A1UA2UA3 U A4)= P[A1]+P[A2]+P[A3] + P(A4)−P[A1∩A2]−P[A1∩A3] P(A1∩A4)−P[A2∩A3]-……..

+ P[A1∩A2∩A3] + P(A1∩A2∩A4) +…….- P(A1∩A2∩A3∩A4)

We want to generalize this formula to the case of n events A1, A2, ... , An; that is, we wish to compute the probability of the event that at least one among the Ak occurs. In symbols this event is

A1 U A2 U….U An

More generally, for n events

A1,A2,...An ∈S we have:

$P_1 =$ P $(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i\ ) - \sum_{i<j}\sum P(A_i\, A_j) + \sum_{i<j<k}\sum\sum P(A_i\, A_j\, A_k\ ) -$ ……..

+ $(-1)^{n+1}$P$(A_1A_2\ldots\ldots A_n)$

Let p_i = P(Ai) p_{ij} = P(Ai ∩ Aj) p_{ijk} = P(Ai ∩Aj ∩Ak)

We know, P $(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i\ ) - \sum_{i<j}\sum P(A_i\, A_j) +$

$\sum_{i<j<k}\sum\sum P(A_i\, A_j\, A_k\ )$ -……..

$+ (-1)^{n+1}P(A_1A_2\ldots\ldots A_n)$ Eq. 1

Let $S1 = \sum_{i=1}^{n} P(A_i)$, $S2 = \sum_{i<j} \sum P(A_i A_j)$,

$S3 = \sum_{i<j<k} \sum \sum P(A_i A_j A_k)$, $Sn = P(A_1A_2\ldots\ldots A_n)$ Eq. 2

Here i < j < k < ... <n, so that in the sums each combination appears once and only once; hence in Sr $there\ are\ {}^{n}C_r$ terms. The last sum, Sn, reduces to the single term P(A1∩A2∩….∩An), which is the probability of the simultaneous realization of all n events. For n = 2 we have only the two terms S1 and S2 and using equations 1 and 2, it can be written

P (A1 U A2) = S1-S2

So, the generalization to an arbitrary number n of events is the Probability of realization of at least one event out of n events and is given by

$P_1 = P(\bigcup_{i=1}^{n} A_i) = S1 - S2 + S3 - \ldots\ldots\ldots\ldots\ldots \pm Sn$ Eq. 3

Probability of Realization of no event out of n events:

P (no event) = 1- P(At least one event) = 1- P_1

P (no event) = 1- (S1 – S2 + S3 - ………….. $\pm Sn$)

= 1- S1 + S2 - S3 - ………….. $\pm Sn$

Probability of Realization of exactly m events out of n events:

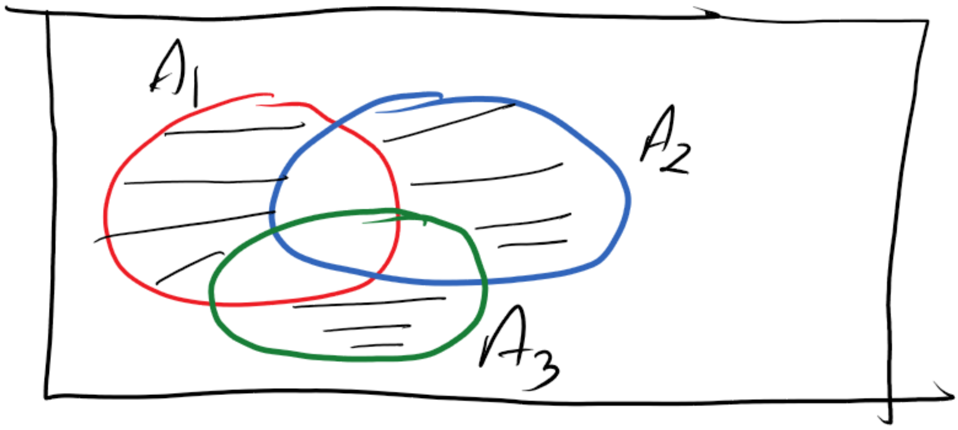
Theorem. For any integer m with 1 < m < n the probability $P_{[m]}$ that exactly m among the n events A_1 .. ,A_n occur simultaneously is given by

$P_{[m]} = S_m - {}^{m+1}C_m\, S_{m+1} + {}^{m+2}C_m\, S_{m+2} - {}^{m+3}C_m\, S_{m+3} +$

$\ldots\ldots \pm {}^{n}C_m\, S_n$ Eq. 4

Proof:

For three events A1, A2, A3, if we find probability of exactly one event, from the Venn diagram we get,



$P_{[1]} =$ P(A1) + P(A2) + P(A3) -2 P(A1A2) – 2 P(A1A3) –

2 P(A2A3) + 3 P(A1A2A3)

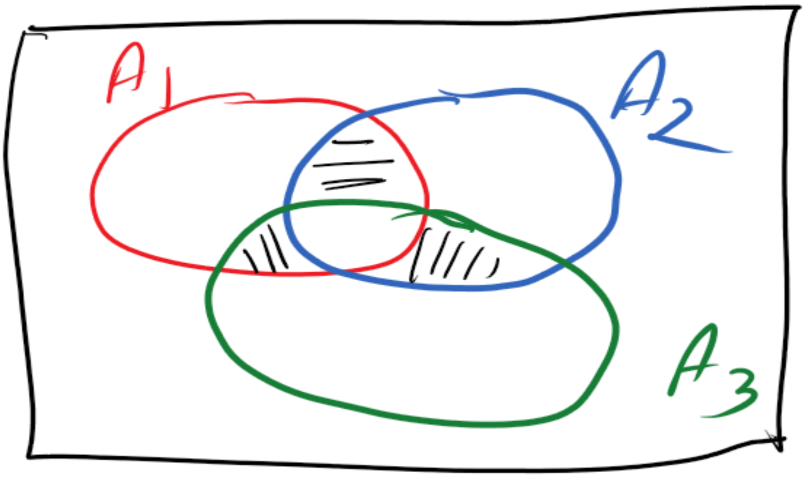
$\Rightarrow P_{[1]} = S_1 - {}^2C_1\, S_2 + {}^3C_1 S_3$

which proves the theorem.

For n events

$P_{[1]} = S_1 - {}^2C_1\, S_2 + {}^3C_1 S_3 - \cdots \ldots \ldots \ldots . \pm {}^nC_1 S_n$ Eq. 5

For three events A1, A2, A3, if we find probability of exactly two events, from the Venn diagram we get,



$P_{[2]} =$ P(A1A2) + P(A1A3) + P(A2A3) - 3 P(A1A2A3)

$\Rightarrow P_{[2]} = S_2 - {}^3C_2S_3$

For n events
$P_{[2]} = S_2 - {}^3C_2\, S_3 + {}^4C_2S_4 - \cdots \ldots \ldots \ldots . \pm {}^nC_2S_n$ Eq. 6

which proves the theorem.

$$P_{[0]} = None\ among\ A_k\ occurs = 1 - p(Atleast\ one\ occurs)$$

= 1- $(S_1 - S_2 + S_3 - - - - - - \pm S_n)$

= $S_0 - S_1 + S_2 - \ S_3 + - - - - - \pm S_n$

= $S_0 - {}^1C_0S_1 + {}^2C_0S_2 - \ {}^3C_0S_3 + - - - - - \pm {}^nC_0S_n$ Eq. 7

which proves the theorem

Probability of Realization of at least m events out of n events:

Theorem. For any integer m with 1 < m < n the probability P_m
that at least m among the n events A_1 .. , A_n occur simultaneously is
given by

P_m = $S_m - \ {}^mC_{m-1}\, S_{m+1} + \ {}^{m+1}C_{m-1}\, S_{m+2} - \ {}^{m+2}C_{m-1}\, S_{m+3} +$

$\ldots . . \pm {}^{n-1}C_{m-1}\, S_n$ Eq. 8

Proof: $P_{m+1} = P_m - \ P_{[m]}$

Now, $P_2 = P_1 - \ P_{[1]}$ Using Eq. 3 and Eq. 5, we get,

$P_2 =$ (S1 – S2 + S3 - S_4………….. ± Sn) –

$(S_1 - {}^2C_1\, S_2 + {}^3C_1S_3 - {}^4C_1S_4 \ldots \ldots \ldots \ldots . \pm {}^nC_1S_n)$

$P_2 =$ (2-1)$S_2 - (3-1)S_3 + (4-1)S_4 \ldots \ldots \ldots \ldots \ldots \pm \ (n-1)S_n$

$P_2 = S_2 - 2S_3 + 3S_3 \ldots \ldots \ldots \ldots \ldots . . \pm (n-1)S_n$

$\Rightarrow P_2 = S_2 - {}^2C_1S_3 + {}^3C_1S_4 - \cdots \ldots \ldots \pm {}^{n-1}C_1S_n$ Eq.9

which proves Eq. 8

Now, $P_3 = P_2 - P_{[2]}$ Using Eq. 6 and Eq. 9, we get,

$$P_3 = (S_2 - {}^2C_1S_3 + {}^3C_1S_4 - \cdots \ldots \ldots \pm {}^{n-1}C_1S_n) -$$

$$(S_2 - {}^3C_2\, S_3 + {}^4C_2S_4 - \cdots \ldots \ldots \ldots . \pm {}^nC_2S_n)$$

$$P_3 = ({}^3C_2 - {}^2C_1)S_3 - ({}^4C_2 - {}^3C_1)S_4 + \cdots \ldots \pm ({}^nC_2 - {}^{n-1}C_1)S_n$$

$$P_3 = S_3 - 3S_4 + \cdots \ldots \ldots \ldots \ldots . . \pm \frac{(n-1)(n-2)}{2} S_n$$

$$\Rightarrow P_3 = S_3 - {}^3C_2S_4 + \cdots . . \ldots \ldots \ldots \pm {}^{n-1}C_2S_n$$

which proves Eq. 8

Classical Occupancy Problem:

Consider the set or "population" of n elements a_1 a_2, ••• , a_n. Any ordered arrangement a_{j1}, a_{j2}, ••• , a_{jr} of r symbols is called an ordered sample of size r drawn from our population. For an intuitive picture we can imagine that the elements are selected one by one. Two procedures are then possible. First, sampling with replacement; here each selection is made from the entire population, ·so that the same element can be drawn more than once. The samples are then arrangements in which repetitions are permitted. Second, sampling without replacement; here an element once chosen is removed from the population, so that the sample becomes an arrangement without repetitions. Obviously, in this case, the sample size r cannot exceed the population size n. In sampling with replacement each of the r elements can be chosen in n ways: the number of possible samples is therefore n^r.

Ex. If a multiple choice exam has 20 questions, each of which has 5 possible answers, then there are 5^{20} different ways of completing the exam.

Theorem.

For a population of n elements and a prescribed sample size r, there exist n^r different samples with replacement and

$(n)_r = n(n-1)\dots(n-r+1)$ samples without replacement.

Ex. Consider selecting 5 cards from 52without replacement.

There are 52.51.50.49.48 possible selections.

We note the special case where r = n. In sampling without replacement a sample of size n includes the whole population and represents a reordering (or permutation) of its elements. Accordingly, n elements $a_1 \dots , a_n$ can be ordered in $(n)_n = n \cdot (n-1) \dots \cdot 2 \cdot 1$ different ways. Instead of (n)n we write n!, which is the more usual notation. We see that our theorem has the following

Corollary. The number of different orderings of n elements is

$n! = n\,(n-1) \dots \cdot 2 \cdot 1$.

(i) The number of distinguishable permutations of n indistinct objects, comprising ni items of type i for i = 1,...,κ is

$$\frac{\boldsymbol{n}!}{\boldsymbol{r1}!.\boldsymbol{r2}!\dots\dots\boldsymbol{rk}!}$$

(ii) The number of distinguishable distributions of putting r indistinguishable balls in n cells is

$$A_{r,n} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

(iii) The number of distinguishable distributions in which no cell remains empty is $\binom{r-1}{n-1}$

Proof:

(i) We represent the balls by stars and indicate the n cells by the n spaces between n + 1 bars. Thus I *** I * I I I I **** I is used as a symbol for a distribution of r = 8 balls in n = 6 cells with occupancy numbers 3, 1, 0, 0, 0, 4.

Such a symbol necessarily starts and ends with a bar, but the remaining n - 1 bars and r stars can appear in an arbitrary order. In this way it becomes apparent that the number of distinguishable distributions equals the number of ways of selecting r places out of n + r – 1.

Thus $A_{r,n} = \binom{n+r-1}{r} = \binom{n+r-1}{n-1}$

(ii) The condition that no cell be empty imposes the restriction that no two bars be adjacent. The r stars leave r - 1 spaces of which n - 1 are to be occupied by bars: thus we have $\binom{r-1}{n-1}$

choices and (ii) is proved

Let us consider the problem of a random distribution of r balls in n cells, as there are n^r arrangements, each arrangement has probability $1/n^r$.

Let the probability Pm(r, n) of finding exactly m cells empty.

Let Ai be the event that cell number i is empty (i = 1,2, ... , n).

In this event all r balls are placed in the remaining n - 1 cells, and this can be done in $(n-1)^r$ different ways. Similarly, there are $(n-2)^r$ different ways by which 2 cells can be kept empty

and so on.

So we get $P(A_i) = \frac{(n-1)^r}{n^r} = (1-\frac{1}{n})^r$

$$P(A_i \cap A_j) = \frac{(n-2)^r}{n^r} = (1-\frac{2}{n})^r$$

$$P(A_i \cap A_j \cap A_k) = \frac{(n-3)^r}{n^r} = (1-\frac{3}{n})^r$$

and so on

$$S1 = \sum_{i=1}^{n} P(A_i\ ) = \binom{n}{1} (1-\frac{1}{n})^r$$

$$S2 = \sum_{i<j} \sum P(A_i\, A_j) = \binom{n}{2} (1-\frac{2}{n})^r$$

$$S3 = \sum_{i<j<k} \sum \sum P(A_i\, A_j\, A_k\ ) = \binom{n}{3} (1-\frac{3}{n})^r$$

and so on

(i) P (at least one cell is empty)

$= P(\bigcup_{i=1}^{n} A_i) = S1 - S2 + S3 - \ldots\ldots\ldots\ldots\ldots \pm Sn$

(ii) P (all cells are occupied)

$= P_0 (r, n) = 1 - S1 + S2 - S3 + \ldots\ldots$

$= 1 - \binom{n}{1} (1-\frac{1}{n})^r + \binom{n}{2} (1-\frac{2}{n})^r - \binom{n}{3} (1-\frac{3}{n})^r + \ldots\ldots.$

$= \sum_{i=0}^{n} (-1)^i \binom{n}{i} (1-\frac{i}{n})^r$

(iii) Consider now a distribution in which exactly m cells are empty. These m cells can be chosen in $\binom{n}{m}$ ways. The r balls are distributed among the remaining n - m cells so that each of these cells is occupied; the number of such distributions is $(n-m)^r p_0 (r, n - m)$. Dividing by n^r we find for the probability that exactly m cells remain empty $p_m(r, n) = \binom{n}{m} (1 - \frac{m}{n})^r p_0 (r, n - m)$

Example 1:

True/False exam has 20 questions. Let E = “16 answers correct at random”. Then

P (E) = (Number of ways of getting 16 out of 20 correct)/(Total number of ways of answering 20 questions)

$= \frac{\binom{20}{16}}{2^{20}} = 0.0046.$

Example 2:

If n balls are randomly placed into n cells, the probability that each cell will be occupied equals $n!/n^n$. It is surprisingly small. For n = 7 it is only 0.00612

Example 3. A die is rolled twice. Let all the elementary events in S = { (i, j): i, j = 1, 2, ... , 6} be assigned the same probability. Let A be the event that the first throw shows a number ≤ 2, and B be the event that the second throw shows at least 5. Then

$A = \{(i, j): 1 \leq i \leq 2, j = 1, 2, \dots, 6\}$,

$B = \{(i, j): 5 \leq j \leq 6, i = 1, 2, \dots, 6\}$,

$A \cap B = \{(1, 5), (1, 6), (2, 5), (2, 6)\}$;

and

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 1/3 + 1/3 - 4/36 = 5/9$

Example 4. A coin is tossed three times. Let us assign equal probability to each of the 8 elementary events in S. Let A be the event that at least one head shows up in three throws. Then

$P(A) = I - P(A^c)$

$= 1 - P(\text{no heads}) = I - P(TTT) = 1 - 1/8 = 7/8$

Example 5: The probabilities of n independent events are p1, p2, ……pn respectively. Find the probability that at least one of the event will occur?

Solution:

P(at least one of the event will occur) = 1 – P(None will occur)

$= 1 - P(A1^c \cap A2^c \cap \ldots\ldots \cap An^c)$

$= 1 - P(A1^c)\, P(A2^c) \ldots\ldots\ldots P(An^c)$

$= 1 - (1\text{-}p1)\,(1\text{-}p2) \ldots\ldots\ldots (1\text{-}pn)$

$= 1 - \prod_{i=1}^{n}(1 - pi)$

Example 6. What is the probability that a Bridge hand will have a complete suit?

Bridge is played between four players. 52 cards are distributed to four players so that each player gets 13 cards.

n (S) = No. of ways of drawing 13 cards from 52 cards.

$= {}^{52}C_{13}$

Let A = event that a player has a complete suit

There are four suits and out of which each player will have a complete suit.

n (A) = ${}^{4}C_{1} \times {}^{13}C_{13}$

P (A) = $\frac{n\ (A)}{n\ (S)} = \frac{{}^{4}C_{1} \times {}^{13}C_{13}}{{}^{52}C_{13}}$

Example 7. The population of 52 cards consists of four classes, each of thirteen elements. The probability that a hand of thirteen cards consists of five spades, four hearts, three diamonds, and one club is

P (A) = $\frac{n(A)}{n(S)} = \frac{{}^{13}C_{5} \times {}^{13}C_{4} \times {}^{13}C_{3} \times {}^{13}C_{1}}{{}^{52}C_{13}}$

Example 8. Consider a class of r students. The birthdays of these r students form a sample of size r from the 365 days in the year. Then the probability that all r birthdays are different is ${}^{365}P_{r}$ / $(365)^r$.

Example 9. Consider a poker hand of 5 cards. Find the probability of having 4 of same kind.

n (S) = ${}^{52}C_5$

n (A) = (${}^{4}C_1 \times {}^{13}C_4 \times {}^{39}C_1$)

P (A) $= \frac{{}^{4}C_1 \times {}^{13}C_4 \times {}^{39}C_1}{{}^{52}C_5}$

Example 10. Find the probability that a poker hand has exactly two kings?

n (S) = ${}^{52}C_5$

n (A) = (${}^{4}C_2 \times {}^{48}C_3$)

P (A) $= \frac{{}^{4}C_2 \times {}^{48}C_3}{{}^{52}C_5}$

Example 11. Find the probability that a poker hand has at least two kings?

n (S) = ${}^{52}C_5$

n (A) = (${}^{4}C_2 \times {}^{48}C_3$) + (${}^{4}C_3 \times {}^{48}C_2$) + (${}^{4}C_4 \times {}^{48}C_1$)

P (A) $= \frac{({}^{4}C_2 \times {}^{48}C_3) + ({}^{4}C_3 \times {}^{48}C_2) + ({}^{4}C_4 \times {}^{48}C_1)}{{}^{52}C_5}$

Example 12. Compute the probability that a hand of 5 cards will have

(i) Exactly three clubs

(ii) Exactly three cards of the same suit

Solution: (i) n (S) = ${}^{52}C_5$

n (A) = (${}^{13}C_3 \times {}^{39}C_2$)

P (A) $= \frac{{}^{13}C_3 \times {}^{39}C_2}{{}^{52}C_5}$

(ii) n (S) = $^{52}C_5$

n (A) = ($^{4}C_1 \times {}^{13}C_3 \times {}^{39}C_2$)

$$P\ (A) = \frac{^{4}C_1 \times {}^{13}C_3 \times {}^{39}C_2}{^{52}C_5}$$

Example 13. A card is chosen at random from a deck of 52 cards. Let A be the event that the card is an ace, and B, the event that it is a club. Check their independence.

Example 14. Consider families with two children, and assume that all four possible distributions of gender: BB, BG, GB, GG, where B stands for boy and G for girl, are equally likely. Let E be the event that a randomly chosen family has at most one girl, and F, the event that the family has children of both genders. Check their independence.

Example: 15

Assume that a certain school has equal number of boys and girls. 5% of boys are football players. Find the probability that randomly selected student is a boy and football player.

Solution: Let B = event that a boy is selected

G = event that a girl is selected

F = event that the student is a football player

P(B) = ½ = 0.5 P(G) = ½ = 0.5 P(F/B) = 0.05

$P(F/B) = \frac{P(F\cap B)}{P(B)} \Rightarrow P(F\cap B) = P(F/B)\ P(B)$

P (randomly selected student is a boy and football player)

$= P(F\cap B) = P(F/B)\ P(B)$

$= 0.05 \times 0.5 = 0.025$

Example: 16

Susan took two tests. The probability of her passing both tests is 0.6. The probability of her passing the first test is 0.8. What is the probability of her passing the second test given that she has passed the first test?

Solution:

Let A = event that Susan passes first test
B = event that she passes the second test

$P(A) = 0.8$ $\qquad$ $P(A \cap B) = 0.6$

*P (*passing the second test given that she has passed the first test)

$$= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.6}{0.8} = 0.75$$

Example: 17
A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

Solution:

Let A = event that First marble was red
B = event that second marble was blue

$P(A) = 0.5$ $\qquad$ $P(A \cap B) = 0.28$

*P (*selecting a blue marble on the second draw, given that the first marble drawn was red)

$$= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0.28}{0.5} = 0.56$$

Example: 18

A problem in Mathematics is given to three students whose chances of solving it are 1/3, 1/4 and 1/5 (i) What is the probability that the problem is solved? (ii) What is the probability that exactly one of them will solve it?

Solution

Let A, B and C be the events of solving problems by each students respectively.

$P(A) = 1/3,$ $P(B) = 1/4$ $P(C) = 1/5$

$P(A') = 1-1/3 = 2/3$ $P(B') = 1-\frac{1}{4} = \frac{3}{4}$ $P(C') = 1-1/5 = 4/5$

(i) P (Problem is solved) = P(At least one solving)

= 1 - P(None solving the problem)

$= 1 - P(A' \cap B' \cap C')$

$= 1 - P(A') \cdot P(B') \cdot P(C')$

$= 1- (2/3)(3/4)(4/5)$

$= 1- 2/5 = 3/5$

(ii) P (exactly one of them will solve it)

$= P(A' \cap B' \cap C) + P(A' \cap B \cap C') + P(A \cap B' \cap C')$

$= P(A')P(B')P(C) + P(A')P(B)P(C') + P(A)P(B')P(C')$

$= (2/3)(3/4)(1/5) + (2/3)(1/4)(4/5) + (1/3)(3/4)(4/5)$

$= (6/60) + (8/60) + (12/60)$

$= (6 + 8 + 12)/60 = 26/60$

P (exactly one of them will solve it) = 13/30

Example: 19

The probability that a car being filled with petrol will also need an oil change is 0.30; the probability that it needs a new oil filter is 0.40; and the probability that both the oil and filter need changing is 0.15.

(i) If the oil had to be changed, what is the probability that a new oil filter is needed?

(ii) If a new oil filter is needed, what is the probability that the oil has to be changed?

Solution

Let A and B be the events of changing oil and new oil filter respectively.

$P(A) = 0.30$, $P(B) = 0.40$, $P(A \cap B) = 0.15$

(i) Here we have to find the probability that a new oil filter is needed, if the oil had to be changed. The event B depends on A.

$P(B/A) = P(A \cap B)/P(A) = 0.15 / 0.30 = 1/2$

(ii) If a new oil filter is needed, what is the probability that the oil has to be changed?

The event A depends on B.

$P(A/B) = P(A \cap B)/P(B) = 0.15 / 0.40 = 3/8 = 0.375$

Example: 20

What is the probability that the total of two dice will be greater than 9, given that the first die is a 5?

Solution:

Let A = first die is 5
Let B = total of two dice is greater than 9
$P(A) = \frac{1}{6}$
Possible outcomes for *A* and *B*: (5, 5), (5, 6)
$P(A \text{ and } B) = \frac{2}{36} = \frac{1}{18}$

P (the total of two dice will be greater than 9, given that the first die is a 5)

= P (B/A) = $\frac{\mathrm{P}\ (\mathrm{A}\cap\mathrm{B})}{P\ (A)}$ = $\frac{1/18}{1/6}$ = 1/3

Example: 21

Compute the probability of having a full house in a poker hand?

Solution:

A full house in poker hand means two cards of same denomination and three of same kind other than which has already appeared.

The no. of ways choosing one denomination from 13 denominations is $^{13}C_1$. There are four cards of each denomination. No. of ways selecting two from four is $^{4}C_2$.

Now there are 12 denominations left and selecting one from 12 is $^{12}C_1$. No. of ways selecting three from four is $^{4}C_3$.

$$\text{P (full house in a poker hand)} = \frac{^{13}C_1 \times {}^{4}C_2 \times {}^{12}C_1 \times {}^{4}C_3}{^{52}C_5}$$

$$= 1.44057 \times 10^{-3}$$

Example: 22

Compute the probability of having two pairs in a poker hand?

Solution:

Two pairs in a poker hand means two cards of same denomination, two of same kind other than which has already appeared, one from the remaining denomination.

The no. of ways choosing one denomination from 13 denominations is $^{13}C_1$. There are four cards of each denomination. No. of ways selecting two from four is $^{4}C_2$.

Now there are 12 denominations left and selecting one from 12 is ${}^{12}C_1$. No. of ways selecting two from four is ${}^{4}C_{2.}$

Now there are 11 denominations left and selecting one from 11 is ${}^{12}C_1$. No. of ways selecting one from four is ${}^{4}C_{1.}$

$$\text{P (two pairs in a poker hand)} = \frac{{}^{13}C_1 \times {}^{4}C_2 \times {}^{12}C_1 \times {}^{4}C_2 \times {}^{11}C_1 \times {}^{4}C_1}{{}^{52}C_5}$$

$$= 0.095078$$

Example: 23

In a five-card poker game, find the probability that a hand will have:
(a) A royal flush (ace, king, queen, jack, and 10 of the same suit).

(b) A straight flush (five cards in a sequence, all of the same suit; ace is high but A, 2, 3, 4, 5 is also a sequence), excluding a royal flush.

Solution: P (Royal Flush) = $\frac{{}^{4}C_1 \times {}^{1}C_1 \times {}^{1}C_1 \times {}^{1}C_1 \times {}^{1}C_1 \times {}^{1}C_1}{{}^{52}C_5}$

(b) We count the number of ways that five cards can be dealt to produce a straight flush. A straight flush consists of five cards in sequence, each card in the same suit. It requires two independent choices to produce a straight flush:

Choose the rank of the lowest card in the hand. For a straight, the lowest card can be an ace, 2, 3, 4, 5, 6, 7, 8, 9, or 10. So, we choose one rank from a set of 10 ranks. The number of ways to do this is ${}^{10}C_1$.

Choose one suit for the hand. There are four suits, from which we choose one. The number of ways to do this is ${}^{4}C_1$.

The number of ways to produce a straight flush is equal to the product of the number of ways to make each independent choice. Therefore,

P (Royal Flush) = $\frac{{}^{4}C_1 \times {}^{10}C_1}{{}^{52}C_5}$ = 40 / 2,598,960 = 0.00001539077169

Example: 24

Consider a hand of five cards in a game of poker.
Let A = {at least 3 cards of spades}, B = {all 5 cards of spades}. Then
Find P (B/A)

Solution: P (A ∩ B)= P{all5 cards of spades}= $\frac{{}^{13}C_5}{{}^{52}C_5}$

P{B / A}= $\frac{\mathrm{P(A \cap B)}}{P(A)}$

$$= \frac{\frac{{}^{13}C_5}{{}^{52}C_5}}{\frac{{}^{13}C_3 \times {}^{39}C_2 + {}^{13}C_4 \times {}^{39}C_1 + {}^{13}C_5 \times {}^{39}C_0}{{}^{52}C_5}} =$$

Example: 25

Take four identical marbles. On the first, write symbols
A 1 A2A3. On each of the other three, write A 1, A2, A3, respectively. Put the four marbles in an urn and draw one at random.
Check their independence.

Let Ei denote the event that the symbol Ai appears on the drawn marble. Then

P(E1) = P(E2) = P(E3) = 2/4 =1/2.

P(E1E2) = P(E1E3) = P(E2E3) = 1/4.

and P(E1E2E3) = 1/4

It follows that although events E1, E2, E3, are not independent, they are pairwise independent.

Note: A1, A2, ... , An ∈ S, with P($\bigcap_{j=1}^{n-1} A_j$) > 0. Then

$P(\bigcap_{j=1}^{n} A_j)$ = P(A1) P(A2/A1) P(A3/$A1 \cap A2$)…..P(An/$\bigcap_{j=1}^{n-1} A_j$)

This is called Chain Rule of Multiplication Theorem.

Matching Problems:

One of the classic problems in probability theory is the “matching problem.” This problem has many variations and dated back to the early 18th century. There are many ways to describe the problem. One such description is the example of matching letters with envelopes. Suppose there are n letters with n matching envelopes (assume that each letter has only one matching envelope). Further suppose that the secretary stuffs the letters randomly into envelopes. What is the probability that every letter is matched correctly, or that no letter is matched correctly or that exactly r letters are stuffed into the correct envelopes?

Suppose that the n letters are numbered 1, 2, 3, …….,n. Let Ai be the event that the ith letter is stuffed into the correct envelop. Then

$P(\cup_{i=1}^{n} A_i)$ is the probability that at least one letter is matched with the correct envelop.

The total number of ways where n letters can be put in n envelops is n(n-1)(n-2)….2.1 = n!

P(Ai) = P (ith letter is stuffed into the correct envelop)

= [(No. of ways by ith letter goes in ith envelope) × (No. of ways by remaining (n-1) letters goes (n-1) envelopes)] / Total no. of cases

$$= \frac{1\times(n-1)!}{n!} = \frac{1}{n}$$

$P(A_i \cap A_j)$ = P(ith and jth letter go to ith and jth envelopes)

$$= \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)}$$

$P(A_i \cap A_j \cap A_k)$ = P(ith, jth and kth letters go to ith, jth and kth envelopes)

$$= \frac{1 \times 1 \times 1 \times (n-3)!}{n!} = \frac{1}{n(n-1)(n-2)}$$

and so on

S1 = $\sum_{i=1}^{n} P(A_i\ ) = \binom{n}{1} \times \frac{1}{n} = n\ \times \frac{1}{n} = 1$

S2 = $\sum_{i<j} \sum P(A_i\, A_j) = \binom{n}{2} \times \frac{1}{n(n-1)} = \frac{1}{2} = \frac{1}{2!}$

S3 = $\sum_{i<j<k} \sum \sum P(A_i\ A_j\ A_k\ ) = \binom{n}{3} \frac{1}{n(n-1)(n-2)}\ = \frac{1}{3!}$

and so on

(i) P (at least one letter matches correctly)

= $P(\bigcup_{i=1}^{n} A_i)$ = S1 – S2 + S3 - ………….. $(-1)^{n+1}\, S$n

$$= 1- \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \ldots \ldots (-1)^{n+1} \frac{1}{n!}$$

$$= \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

(ii) P (no match) = 1 – P (at least one match)

$$= 1 - \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

$$= \frac{1}{1!} - \ \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k!}$$

$$= \sum_{k=0}^{n} (-1)^{k} \frac{1}{k!}$$

(iii) P (exactly r matches)

= Probability that each of r letters goes in correct envelopes and the remaining (n-r) doesn’t go to correct envelopes.

Now, P (each of r letters go to correct places)

$$= \frac{1\times1\times\ldots.\times1\times(n-r)!}{n!} = \frac{1}{n(n-1)(n-2)\ldots\ldots(n-r+1)}$$

P (none of the remaining (n-r) letters goes in the correct envelope)

$= \sum_{k=0}^{(n-r)} (-1)^k \frac{1}{k!}$

P (exactly r matches)

$= \frac{1}{n(n-1)(n-2)\ldots\ldots(n-r+1)} \sum_{k=0}^{n-r} (-1)^k \frac{1}{k!}$

(iv) Compute the probability of no match and at least one match if the number of letters and envelopes are very large.

P (no match) = $\sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} = e^{-1} = 0.36787$

P (at least one match) = 1 - e^{-1}

$[e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \ldots \ldots]$

Two similar decks of N distinct cards are matched simultaneously against a similar target deck. Find the probability of exactly double matches.

Solution: Let A_i = event that a double match occurs in ith position.

i = 1, 2, ,……..N

The card from the first deck and the card from the second deck should match the target deck.

P (A_i) = P (a double match occurs in ith position)

$$= \frac{1!\times(N-1)!}{N!} \times \frac{1!\times(N-1)!}{N!} = \left(\frac{(N-1)!}{N!}\right)^2$$

Let A_i A_j = event that a double match occurs in ith and jth position.

$i \neq j = 1, 2, ,\ldots\ldots..N$

P $(A_i\ A_j)$ = P (a double match occurs in ith and jth position)

$$= \frac{1!\times 1!\times (N-2)!}{N!} \times \frac{1!\times 1!\times (N-2)!}{N!} = \left(\frac{(N-2)!}{N!}\right)^2$$

$$\text{S1} = \sum_{i=1}^{N} P(A_i\ ) = \binom{N}{1} \times \left(\frac{(N-1)!}{N!}\right)^2$$

$$\text{S2} = \sum_{i<j} \sum P(A_i\ A_j) = \binom{N}{2} \times \left(\frac{(N-2)!}{N!}\right)^2$$

$$\text{S3} = \sum_{i<j<k} \sum \sum P(A_i\ A_j\ A_k\ ) = \binom{N}{3} \left(\frac{(N-3)!}{N!}\right)^2$$

and so on

In general Sm = $\binom{N}{m} \times \left(\frac{(N-m)!}{N!}\right)^2$

$$= \frac{N!}{m!\times (N-m)!} \left(\frac{(N-m)!}{N!}\right)^2 = \frac{(N-m)!}{m!\times N!}$$

P (exactly double matches)

$$= P_{[m]} = S_m - {}^{m+1}C_m\, S_{m+1} + {}^{m+2}C_m\, S_{m+2} - {}^{m+3}C_m\, S_{m+3} + \ldots.. \pm {}^{N}C_m\, S_N$$

$$= S_m - {}^{m+1}C_m\, S_{m+1} + {}^{m+2}C_m\, S_{m+2} - {}^{m+3}C_m\, S_{m+3} + \ldots.. \pm {}^{N}C_m\, S_N$$

$$= \frac{(N-m)!}{m!\times N!} - {}^{m+1}C_m \frac{(N-m-1)!}{(m+1)!\times N!} + {}^{m+2}C_m \frac{(N-m-2)!}{(m+2)!\times N!} - \ldots\ldots\ldots \pm {}^{N}C_m \frac{(N-N)!}{N!\times N!}$$

$$= \sum_{k=0}^{N-m} (-1)^k\, {}^{m+k}C_m \frac{(N-m-k)!}{(m+k)!\times N!}$$

SUBPOPULATIONS AND PARTITIONS

As before, we use the term population of size n to denote an aggregate of n elements without regard to their order. Two populations are considered different only if one contains an element not contained in the other. Consider a subpopulation of size r of a given population consisting of n elements. An arbitrary numbering of the elements of the subpopulation changes it into an ordered sample of size r and, conversely, every such sample can be obtained in this way. Since r elements can be numbered in r! different ways, it follows that there are exactly r! times as many samples as there are subpopulations of size r. The number of subpopulations of size r is therefore given by (n)r/r!. Expressions of this kind are known as binomial coefficients, and the standard notation for them is

$$\binom{n}{r} = \frac{(n)r}{r!} \quad = \frac{n(n-1) \quad (n-r+1)}{1.2\ldots r\ldots.}$$

We have now proved **Theorem 1. A population of n elements possesses $\boldsymbol{\binom{n}{r}}$ different subpopulations of size r $\leq \boldsymbol{n}$.**

The distinction between distinguishable and indistinguishable elements has similarities to the relationship between a subpopulation and the corresponding ordered samples. Deleting all subscripts in an arrangement (or grouping) of 1 elements a1,"" ar yields an at arrangement of f indistinguishable letters. Conversely, an arbitrary numbering bf::the r letters in an arrangement of the latter kind produces an arrangement of the letters aI , ... , ar • This procedure yields r! different arrangements provided, of course, that any interchange of aj and ak counts as rearrangement. The following examples show how this prinCiple' can be applied and extended to situations in which the elements ak,: are only partially identified.

Theorem 2. Let r1, ••• ,rk be integers such that

n = r1+r2+…..+rk

The number of ways in which a population of n elements can be divided into k ordered parts (partitioned into k sub populations) of which the first contains· r1 elements, the second r2 elements, etc., is

$$\frac{\boldsymbol{n}!}{\boldsymbol{r1}!.\boldsymbol{r2}!\ldots\ldots\boldsymbol{rk}!}$$

Examples:

•Dice: A throw of twelve dice can result in 6^{12} different outcomes, total of which we attribute equal probabilities. The event that each face appears twice can occur in as many ways as twelve dice can be arrange in six groups of two each 2+2+⋯+2=12. Hence the probability of the event is

$$(12!/(2!)^{6})/6^{12} = 0.003438....$$

•Bridge: At a bridge table the 52 cards are partitioned in to four equal groups and therefore the number of different situations is $52!/(13!)^{4} = (5.36...)\cdot 1028$. Let us now calculate the probability that each players has an ace. The four aces can be ordered in 4! = 24 ways, and each order represents one possibility of giving one ace to each player. The remaining 48 cards can be distributed in $48!/(12!)^{4}$ ways. Hence the required probability is

$$\frac{24\times(48)!\times 13^{4}}{52!} = 0.105...$$