Paper I

Distribution Theory and Stochastic Process

Unit II: Generating Functions

Introduction: As usual, our starting point is a random experiment modelled by a probability space (Ω, F, P)

. A *generating function* of a real-valued random variable is an expected value of a certain transformation of the random variable involving another (deterministic) variable. Most generating functions share four important properties:

1. Under mild conditions, the generating function completely determines the distribution of the random variable.
2. The generating function of a sum of independent variables is the product of the generating functions
3. The moments of the random variable can be obtained from the derivatives of the generating function.
4. Ordinary (pointwise) convergence of a sequence of generating functions corresponds to the special convergence of the corresponding distributions.

PGFs are useful tools for dealing with sums and limits of random variables. For some stochastic processes, they also have a special role in telling us whether a process will ever reach a particular state. We should be able to:

• find the sum of Geometric, Binomial, and Exponential series;

• know the definition of the PGF, and use it to calculate the mean, variance, and probabilities;

• calculate the PGF for Geometric, Binomial, and Poisson distributions;

• calculate the PGF for a randomly stopped sum;

• calculate the PGF for first reaching times in the random walk;

• use the PGF to determine whether a process will ever reach a given state.

1. Geometric Series

$$1 + r + r^2 + r^3 + \dots \quad = \quad \sum_{x=0}^{\infty} r^x \quad = \quad \frac{1}{1-r}, \qquad \text{when } |r| < 1.$$

2. Binomial Theorem *For any $p, q \in \mathbb{R}$, and integer n,*

$$(p+q)^n \quad = \quad \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x}.$$

3. Exponential Power Series

$$\text{For any } \lambda \in \mathbb{R}, \qquad \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{\lambda}.$$

Probability Generating Functions

The probability generating function (PGF) is a useful tool for dealing with discrete random variables taking values 0, 1, 2, ... Its particular strength is that it gives us an easy way of characterizing the distribution of X +Y when X and Y are independent. In general it is difficult to find the distribution of a sum using the traditional probability function.

Sums of random variables are particularly important in the study of stochastic processes.

The PGF can be used to generate all the probabilities of the distribution. This is generally tedious and is not often an efficient way of calculating probabilities. However, the fact that it can be done demonstrates that the PGF tells us everything there is to know about the distribution.

Definition: Let X be a discrete random variable taking values in the non-negative integers {0, 1, 2, . . .}. The probability generating function (PGF) of X is

GX(s) = E(s^X), for all s ∈ R for which the sum converges.

Calculating the probability generating function

$$G_X(s) = \mathbb{E}\left(s^X\right) = \sum_{x=0}^{\infty} s^x \mathbb{P}(X = x).$$

= $s^0 P(X = 0) + s^1 P(X = 1\ ) + s^2 P(X = 2) + s^3 P(X = 3) + \cdots \ldots$

= $p_0 + sp_1 + s^2 p_2 + s^3 p_3 + \cdots \ldots \ldots ..$

Properties of Probability Generating Functions

1. It is to be noted that $G_X(0)$ =

=$0^0 P(X = 0) + 0^1 P(X = 1\ ) + 0^2 P(X = 2) + 0^3 P(X = 3) + \cdots \ldots$

= 1×P(X = 0) =P(X=0) = p_0

2. $G_X(1)$ = $1^0 P(X = 0) + 1^1 P(X = 1\ ) + 1^2 P(X = 2) + 1^3 P(X = 3) + \cdots \ldots$

P(X = 0) + P(X = 1) + P(X = 2) + · · · = 1

3. Next, consider $G_X(s)$ together with its first and second derivatives $G_X'(s)$ and $G_X''(s)$ (the differentiation is with respect to s=0.):

$G_X'(s)$ = $p_1 + 2s\, p_2 + 3s^2\, p_3 + \cdots \ldots ..$

$G_X'(0)$ = $p_1 + 2 \times 0 \times\ p_2 + 3 \times 0 \times\ p_3 + \cdots \ldots ..$

⇒ $G_X'(0)$ = p_1 = P(X=1)

$G_X''(s)$ = $2\, p_2 + 3\ \times 2 \times s \times\ p_3 + 4\ \times 3 \times s^2 p_4 + \cdots \ldots ..$

= $2\, p_2 + 6\, s\, p_3 + \cdots \ldots ..$

$G_X''(0)$ = $2\, p_2$

$\Rightarrow G_X''(0) = 2p_2 = 2\ P(X=2)$

$\Rightarrow p_2 = P(X = 2) = \frac{1}{2}\ G_X''(0) = \frac{G_X''(0)}{2!}$

$G_X'''(s) = 3\ \times 2 \times\ p_3 + 4 \times 3 \times 2 \times s \times p_4 + \cdots \ldots ..$

$G_X'''(0) = 3\ \times 2 \times\ p_3$

$\Rightarrow p_3 = P(X = 3) = \frac{1}{6}\ G_X'''(0) = \frac{G_X'''(0)}{3!}$

In general:

$$p_n = \mathbb{P}(X = n) = \left(\frac{1}{n!}\right) G_X^{(n)}(0) = \left(\frac{1}{n!}\right) \frac{d^n}{ds^n}(G_X(s))\Big|_{s=0}.$$

4. Uniqueness of the PGF

The formula $p_n = P(X = n) = = \frac{G_X^{(n)}(0)}{n!}$

shows that the whole sequence of probabilities p0, p1, p2, . . . is determined by the values of the PGF and its derivatives at s = 0. It follows that the PGF specifies a unique set of probabilities.

5. Mean and Variance of a Random variable using PGF

$G_X'(s) = p_1 + 2s\, p_2 + 3s^2\, p_3 + 4s^4 p_4 \ldots \ldots ..$

$G_X'(1) = p_1 + 2 \times 1 \times\ p_2 + 3 \times 1 \times\ p_3 + 4\ \times 1 \times p_4 \ldots \ldots ..$

$\Rightarrow\ G_X'(1) = 0 \times P(X = 0) + 1 \times P(X = 1) + 2\ P(X = 2) + 3P(X = 3) + 4\ P(X = 4) + \cdot \cdot \cdot \ldots.$

$\Rightarrow G_X'(1) = \sum_{x=0}^{\infty} xP(X = x)$ = E(X) = Mean of X

$G_X''(s) = 2\, p_2 + 3\ \times 2 \times s \times\ p_3 + 4\ \times 3 \times s^2 p_4 + \cdots \ldots ..$

$= 2\,p_2 + 6\,s\,p_3 + 12s^2 p_4 \ldots \ldots$

$G''(1) = 2.1P(X = 2) + 3.2P(X = 3) + 4.3\,P(X = 4) + \cdots$

$= 2.(2-1)\,P(X = 2) + 3.(3-2)\,P(X = 3) + 4.(4-3)\,P(X = 4) + \cdots$

$= 0.(0-1)\,P(X=0) + 1.(1-1)\,P(X=1) + 2.(2-1)\,P(X = 2) + 3.(3-2)\,P(X = 3)$

$+ 4.(4-3)\,P(X = 4) + \cdots$

$= \sum_{x=0}^{\infty} x(x-1)P(X = x)$

$\Rightarrow G_X''(1) = E[X(X-1)]$

$V(X) = E(X^2) - [E(X)]^2$

$= E[X(X-1)] + E(X) - [E(X)]^2$

So, $V(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2$

6. Suppose that $X_1, \ldots, X_n$ are independent random variables, and let $Y = X_1 + \ldots + X_n$. Then

$$G_Y(s) = \prod_{i=1}^{n} G_{X_i}(s).$$

Proof:

$$\begin{aligned} G_Y(s) &= \mathbb{E}(s^{(X_1+\ldots+X_n)}) \\ &= \mathbb{E}(s^{X_1}s^{X_2}\ldots s^{X_n}) \\ &= \mathbb{E}(s^{X_1})\mathbb{E}(s^{X_2})\ldots\mathbb{E}(s^{X_n}) \\ &\qquad \text{(because } X_1,\ldots,X_n \text{ are independent)} \\ &= \prod_{i=1}^{n} G_{X_i}(s). \quad \text{as required.} \quad \square \end{aligned}$$

Example: Suppose that X and Y are independent with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Find the distribution of $X+Y$.

Solution:

$$\begin{aligned} G_{X+Y}(s) &= G_X(s) \cdot G_Y(s) \\ &= e^{\lambda(s-1)} e^{\mu(s-1)} \\ &= e^{(\lambda+\mu)(s-1)}. \end{aligned}$$

But this is the PGF of the Poisson$(\lambda+\mu)$ *distribution. So, by the uniqueness of PGFs,* $X+Y \sim$ *Poisson*$(\lambda+\mu)$.

Example 1: Binomial Distribution

Let $X \sim \text{Binomial}(n,p)$, so $\mathbb{P}(X=x) = \binom{n}{x} p^x q^{n-x}$ for $x = 0, 1, \ldots, n$.

$$\begin{aligned} G_X(s) &= \sum_{x=0}^{n} s^x \binom{n}{x} p^x q^{n-x} \\ &= \sum_{x=0}^{n} \binom{n}{x} (ps)^x q^{n-x} \\ &= (ps+q)^n \quad \text{by the Binomial Theorem: true for all } s. \end{aligned}$$

Thus $G_X(s) = (ps+q)^n$ *for all* $s \in \mathbb{R}$.

$$G'_X(s) = \frac{d}{d(ps+q)}(ps+q)^n \times \frac{d(ps+q)}{ds}$$

$$= n\,(ps+q)^{n-1}\,p$$

$$G''_X(s) = \text{n p}\ \frac{d}{d(ps+q)}(ps+q)^{n-1} \times \frac{d(ps+q)}{ds}$$

$$np\,(n-1)(ps+q)^{n-2}\,p$$

Accordingly:

$$G(1) = (q+p)^n = 1^n = 1$$
$$G'(1) = n(q+p)^{n-1}p = n.1^{n-1}p = np$$
$$G''(1) = n(n-1)(q+p)^{n-2}p^2 = n(n-1).1^{n-2}p^2 = n(n-1)p^2$$

By (6.2), $G'(1) = \mathrm{E}(X) = np$. This result was derived very much more tediously on page 5.1.

The variance is derived from (6.3):

$$\begin{aligned} \mathrm{V}(X) &= G''(1) + G'(1) - \left(G'(1)\right)^2 \\ &= n(n-1)p^2 + np - (np)^2 \\ &= n^2p^2 - np^2 + np - n^2p^2 \\ &= np(1-p) \\ &= npq \end{aligned}$$

Example 2: Poisson Distribution

Let $X \sim \text{Poisson}(\lambda)$, so $\mathbb{P}(X = x) = \frac{\lambda^x}{x!}e^{-\lambda}$ for $x = 0, 1, 2, \ldots$.

$$G_X(s) = \sum_{x=0}^{\infty} s^x \frac{\lambda^x}{x!}e^{-\lambda} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(\lambda s)^x}{x!}$$

$$= e^{-\lambda}e^{(\lambda s)} \quad \text{for all } s \in \mathbb{R}.$$

$$= e^{\lambda(s-1)}$$

$$G'_X(s) = \frac{d}{ds}\left[e^{-\lambda}\, e^{\lambda s}\right]$$
$$= e^{-\lambda}\left[\lambda e^{\lambda s}\right]$$
$$G''_X(s) = e^{-\lambda}[\lambda \times \lambda \times e^{\lambda s}] = \lambda^2\, e^{-\lambda}\, e^{\lambda s}$$

Accordingly:

$$G(1) = e^{\lambda} e^{-\lambda} = 1$$
$$G'(1) = \lambda e^{\lambda} e^{-\lambda} = \lambda$$
$$G''(1) = \lambda^2 e^{\lambda} e^{-\lambda} = \lambda^2$$

By (6.2), $\mathrm{E}(X) = \lambda$.

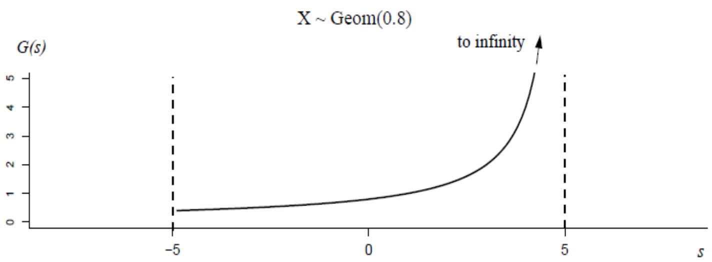
The variance is derived from (6.3):

$$\begin{aligned} \mathrm{V}(X) &= G''(1) + G'(1) - \left(G'(1)\right)^2 \\ &= \lambda^2 + \lambda - \lambda^2 \\ &= \lambda \end{aligned}$$

Example 3: **Geometric Distribution**

Let $X \sim \text{Geometric}(p)$, so $\mathbb{P}(X = x) = p(1-p)^x = pq^x$ for $x = 0, 1, 2, \ldots$, where $q = 1 - p$.

$$\begin{aligned} G_X(s) &= \sum_{x=0}^{\infty} s^x p q^x \\ &= p \sum_{x=0}^{\infty} (qs)^x \\ &= \frac{p}{1-qs} \quad \text{for all } s \text{ such that } |qs| < 1. \end{aligned}$$



Thus $\quad G_X(s) = \dfrac{p}{1-qs} \quad$ *for* $\quad |s| < \dfrac{1}{q}$.

$$G_X'(s) = \frac{d}{d(1-qs)}\, p\,(1-qs)^{-1} \times \frac{d\,(1-qs)}{ds}$$
$$= p(-1)\,(1-qs)^{-2}\,(-q) \; = pq\,(1-qs)^{-2}$$
$$G_X''(s) = \text{p q}\; \frac{d}{d(1-qs)} (1-qs)^{-2} \times \frac{d(1-qs)}{ds}$$
$$= pq\,(-2)\,(1-qs)^{-3}\,(-q) \; = 2\,pq^2\,(1-qs)^{-3}$$

Accordingly:

$$G(1) = p(1-q)^{-1} = p.p^{-1} = 1$$
$$G'(1) = p(1-q)^{-2}q = \frac{p}{p^2}q = \frac{q}{p}$$
$$G''(1) = 2p(1-q)^{-3}q^2 = 2\frac{p}{p^3}q^2 = 2\frac{q^2}{p^2}$$

By (6.2), $\mathrm{E}(X) = \frac{q}{p}$.

The variance is derived from (6.3):

$$\begin{aligned} \mathrm{V}(X) &= G''(1) + G'(1) - \left(G'(1)\right)^2 \\ &= 2\frac{q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} \\ &= \frac{q}{p}\left(\frac{q}{p} + 1\right) \\ &= \frac{q}{p}\left(\frac{q+p}{p}\right) \\ &= \frac{q}{p^2} \end{aligned}$$

Theorem 1: If X is a r.v. which assumes only integral values with probability distribution: P(X=k) = $p_k; k = 0, 1, 2, \ldots \ldots.$ *and*

$$P(X > k) = q_k, \qquad k \geq 0 \text{ so that } q_k = p_{k+1} + p_{k+2} + \cdots \ldots.. = 1 - \sum_{i=0}^{k} p_i \text{ and two generating functions are :}$$

$$G_X(s) = p_0 + sp_1 + s^2p_2 + s^3p_3 + \cdots \ldots \ldots ..$$

And $Q_X(s) = q_0 + sq_1 + s^2q_2 + s^3q_3 + \cdots \ldots \ldots ..$

Then for -1<s<1, $Q_X(s) = \frac{1-G_X(s)}{1-s}$

Proof: Since $P(X > k) = q_k$, we get $P(X > k-1) = q_{k-1}$

p_k = P(X=k) = $P(X > k-1) - P(X > k) = q_{k-1} - q_k$; k≥ 1

$\Rightarrow \sum_{k=1}^{\infty} s^k q_{k-1}$ - $\sum_{k=1}^{\infty} s^k q_k = \sum_{k=1}^{\infty} s^k p_k$

$\Rightarrow (sq_0 + s^2q_1 + s^3q_2 + s^4q_3 + \cdots \dots \dots ..) - [(q_0 + sq_1 + s^2q_2 + s^3q_3 + \cdots \dots \dots ..) + q_0] = [(p_0 + sp_1 + s^2p_2 + s^3p_3 + \cdots \dots \dots ..) - p_0]$

$\Rightarrow$ s $(q_0 + sq_1 + s^2q_2 + s^3q_3 + \cdots \dots \dots ..)$ - $Q_X(s) + q_0 = G_X(s) - p_0$

$\Rightarrow$ s $Q_X(s)$ - $Q_X(s) + q_0 = G_X(s) - p_0$

$\Rightarrow Q_X(s)(s-1) = G_X(s) - p_0 - q_0$

$\Rightarrow Q_X(s) = \frac{G_X(s)-(p_0+q_0)}{s-1}$

$\Rightarrow Q_X(s) = \frac{(p_0+q_0)-G_X(s)}{1-s}$

$\Rightarrow Q_X(s) = \frac{1-G_X(s)}{1-s}$

[Since $p_0 + q_0 = p_0 + p_1 + p_2 + \cdots \dots \dots = 1$]

Theorem 2: If X is a r.v. which assumes only integral values with probability distribution: P(X=k) = $\boldsymbol{p_k; k = 0, 1, 2, \dots \dots . and}$

$$\boldsymbol{P(X > k) = q_k, \quad k \geq 0 \; so \; that \; q_k = p_{k+1} + p_{k+2} + \cdots \dots .. = 1 - \sum_{i=0}^{k} p_i \; and \; two \; generating \; functions \; are:}$$

$\boldsymbol{G_X(s) = p_0 + sp_1 + s^2p_2 + s^3p_3 + \cdots \dots \dots ..}$

And $\boldsymbol{Q_X(s) = q_0 + sq_1 + s^2q_2 + s^3q_3 + \cdots \dots \dots ..}$

E(X) = $\boldsymbol{Q_X(1) and \; V(X) = 2Q'_X(1) + Q_X(1) - \{Q_X(1)\}^2}$

Proof: $\Rightarrow Q_X(s) = \frac{1-G_X(s)}{1-s}$

$\Rightarrow Q_X(s)(1-s) = 1 - G_X(s)$

Differentiating both sides w.r.t s we get,

$Q'_X(s)\,(1-s) + Q_X(s)(-1) = -G'_X(s)$ Eq. 1

Differentiating Eq. 1 both sides w.r.t s we get,

$$\Rightarrow Q'_X(s)\frac{d}{ds}(1-s) + (1-s)\left(\frac{d}{ds}Q'_X(s)\right) - Q'_X(s) = -G''_X(s)$$

$\Rightarrow Q'_X(s)\,(-1) + (1-s)Q''_X(s) - Q'_X(s) = -G''_X(s)$

$\Rightarrow -2\, Q'_X(s) + (1-s)Q''_X\,(s) = -G''_X(s)$

$\Rightarrow G''_X(s) = 2Q'_X\,(s) + (s-1)Q''_X(s)$ Eq. 2

Since $\Rightarrow G'_X\,(1) = \sum_{x=0}^{\infty} xP(X=x)$ = E(X) = Mean of X

In eq. 1 substituting s=1, we get

$Q'_X(1)\,(1-1) + Q_X(1)(-1) = -G'_X(1)$

$\Rightarrow E(X) = G'_X(1) = Q_X(1)$

In eq. 2 substituting s=1, we get

$G''_X(1) = 2Q'_X\,(1) + (1-1)Q''_X(1)$

$\Rightarrow G''_X(1) = 2Q'_X(1)$

Since V (X) = $G''_X\,(1) + G'_X\,(1) - [G'_X\,(1)]^2$

= $2Q'_X(1) + Q_X(1) - \left(Q_X(1)\right)^2$

Convolution:

This definition arises from multiplying polynomials; if $f(x) = \sum_{m=0}^{\infty} a_m x^m$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$, then assuming everything converges we have

$$h(x) = f(x)g(x) = \sum_{k=0}^{\infty} c_k x^k,$$

with $c = a * b$. For example, if $f(x) = 2 + 3x - 4x^2$ and $g(x) = 5 - x + x^3$, then $f(x)g(x) = 10 + 13x - 23x^2 + 6x^3 + 3x^4 - 4x^5$. According to our definition, c_2 should equal

$$a_0 b_2 + a_1 b_1 + a_2 b_0 = 2 \cdot 0 + 3 \cdot (-1) + (-4) \cdot 5 = -23,$$

which is exactly what we get from multiplying $f(x)$ and $g(x)$.

Where

$$c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_{k-1} b_1 + a_k b_0 = \sum_{\ell=0}^{k} a_\ell b_{k-\ell}.$$

We frequently write this as $c = a * b$.

Theorem

Let $\{a_m\}$ *and* $\{b_n\}$ are two sequences with probability generating functions $G_X(s)$ *and* $R_X(s)$ respectively and $W_X(s)$ is their convolution, then $W_X(s) = G_X(s)\, R_X(s)$ where $W_X(s) = \sum_{k=0}^{\infty} c_k s^k$ is the probability Generating Function of Z= X+Y

Proof: $G_X(s) = \sum_{m=0}^{\infty} P(X = m)s^m = \sum_{m=0}^{\infty} a_m s^m$

$R_X(s) = \sum_{n=0}^{\infty} P(X = n)s^n = \sum_{n=0}^{\infty} b_n s^n$

From definition of convolution

$\Rightarrow W_X(s) = G_X(s)\, R_X(s) = \sum_{k=0}^{\infty} c_k s^k$

Where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots \ldots \ldots + a_k b_0$

But we also know that $W_X(s) = G_X(s)\, R_X(s)$ **is the PGF of Z= X+Y**

Relation between Bernoulli and Binomial distributions in terms of convolutions:

The convolution of two independent identically distributed Bernoulli random variables is a binomial random variable.

Let X_1, X_2 ~Bernoulli (p)

$f(x_1) = p^{x_1}(1-p)^{(1-x_1)}, \quad f(x_2) = p^{x_2}(1-p)^{1-x_2}$

Where $x_1 = 0, 1$ *and* $x_2 = 0, 1$

Let $G_X(s) = PGF\ of\ X_1 = \sum_{m=0}^{\infty} P(X_1 = m)s^m = \sum_{m=0}^{1} P(X_1 = m)s^m$

$= P(X_1 = 0)s^0 + P(X_1 = 1)s^1$

= (1-p) + ps

Similarly, $R_X(s) = PGF\ of\ X_2 = (1-p) + ps$

$W_X(s) = G_X(s)\, R_X(s)$ = **PGF of** $X_1 + X_2$

$\Rightarrow W_X(s) = \left((1-p) + ps\right)^2$

$= (q + ps)^2$

Now PGF of Binomial (n, p) distribution is $(ps+q)^n$

Now PGF of Binomial (2, p) distribution is $(ps+q)^2$

So convolution of Bernoulli distribution is Binomial distribution.

Example: Suppose that X and Y are independent with $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Find the distribution of $X+Y$.

Solution:

$$\begin{aligned} G_{X+Y}(s) &= G_X(s) \cdot G_Y(s) \\ &= e^{\lambda(s-1)} e^{\mu(s-1)} \\ &= e^{(\lambda+\mu)(s-1)}. \end{aligned}$$

But this is the PGF of the Poisson$(\lambda+\mu)$ distribution. So, by the uniqueness of PGFs, $X+Y \sim$ Poisson$(\lambda+\mu)$.

Relation between Geometric and Negative Binomial distributions in terms of convolutions:

The convolution of two independent identically distributed Geometric is a Negative binomial random variable.

Let X_1, X_2 ~Geometric (p)

$$f(x_1) = p(1-p)^{(x_1)}, \quad f(x_2) = p(1-p)^{x_2}$$

Let $G_X(s) = PGF\ of\ X_1 = \sum_{m=0}^{\infty} P(X_1 = m)s^m = \frac{p}{1-qs}$

$R_X(s) = PGF\ of\ X_2 = \sum_{n=0}^{\infty} P(X_2 = n)s^n = \frac{p}{1-qs}$

$\boldsymbol{W_X(s) = G_X(s)\, R_X(s)}$ **= PGF of $\boldsymbol{X_1 + X_2}$**

$\Rightarrow W_X(s) = \left(\frac{p}{1-qs}\right)^2$

If $Y = X_1 + X_2 + \cdots .. + X_n$

$W_Y(s) = W_{X_1+X_2+\cdots..X_n}(s) = \left(\frac{p}{1-qs}\right)^n$

For Negative Binomial Distribution,

$f(x) = {}^{x+k-1}C_x p^k q^x$ x= 0, 1, 2, …..

Here p and k are parameters. k = I, 2, 3, ... and 0 < p < 1

(q = 1 – p)

$$G_X(s) = E(s^X) = \sum_{x=0}^{\infty} s^x \left({}^{x+k-1}C_x p^k q^x\right)$$

$$= \sum_{x=0}^{\infty} {}^{x+k-1}C_x p^k (sq)^x$$

$$= \left(\frac{p}{1-qs}\right)^k$$

For k=2, $G_X(s) = \left(\frac{p}{1-qs}\right)^2$

Similarly, for k=n, $G_X(s) = \left(\frac{p}{1-qs}\right)^n$

So Negative Binomial Distribution is n fold convolution of Geometric Distribution.