# Practical 5 - Objectives

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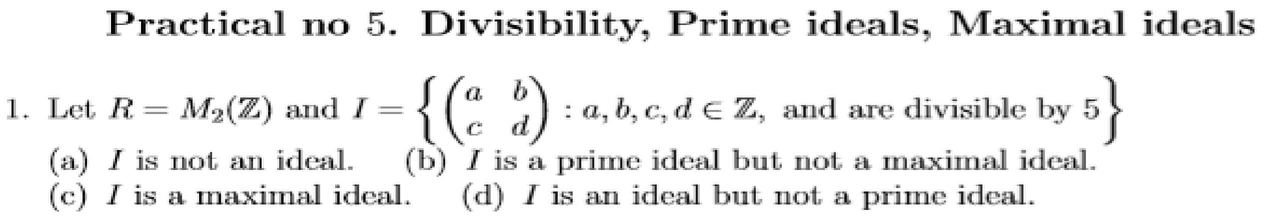
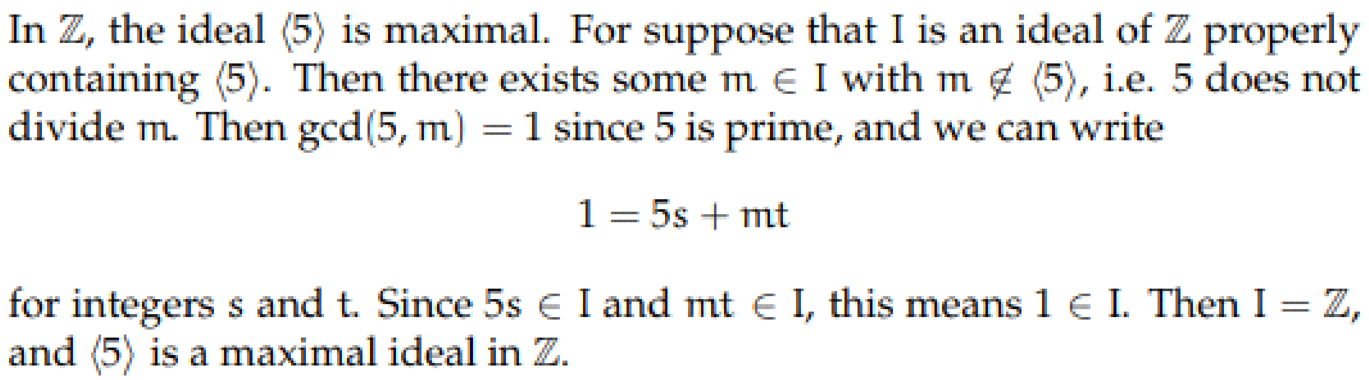


## Practical no 5. Divisibility, Prime ideals, Maximal ideals

1. Let $R = M_2(\mathbb{Z})$ and $I = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, \text{ and are divisible by } 5\right\}$
   (a) $I$ is not an ideal. (b) $I$ is a prime ideal but not a maximal ideal.
   (c) $I$ is a maximal ideal. (d) $I$ is an ideal but not a prime ideal.

**Solution :-** Here , I is an ideal of $\mathbb{M}_2(\mathbb{Z})$

Now if we tahe R = $\mathbb{Z}$ then J = 5 $\mathbb{Z}$ is the maximal ideal of R as



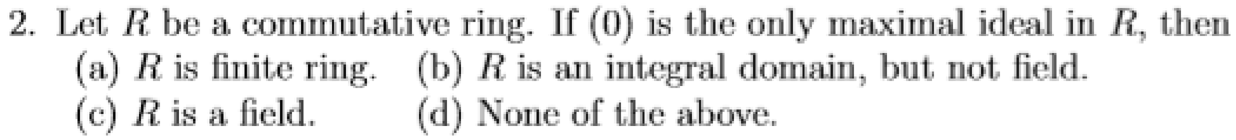
In $\mathbb{Z}$, the ideal $\langle 5\rangle$ is maximal. For suppose that I is an ideal of $\mathbb{Z}$ properly containing $\langle 5\rangle$. Then there exists some $m \in I$ with $m \notin \langle 5\rangle$, i.e. 5 does not divide m. Then $\gcd(5, m) = 1$ since 5 is prime, and we can write

$$1 = 5s + mt$$

for integers s and t. Since $5s \in I$ and $mt \in I$, this means $1 \in I$. Then $I = \mathbb{Z}$, and $\langle 5\rangle$ is a maximal ideal in $\mathbb{Z}$.

$\mathbb{M}_2(\mathbb{Z})$ / $\mathbb{M}_2(5\,\mathbb{Z})$ is an field henve I is an maximal ideal

Option (c)



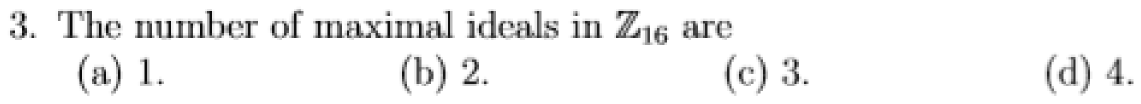
2. Let $R$ be a commutative ring. If (0) is the only maximal ideal in $R$, then
   (a) $R$ is finite ring. (b) $R$ is an integral domain, but not field.
   (c) $R$ is a field. (d) None of the above.

**Solution :-** Theorem 5 states that

5. Show that a field has no ideals except 0 and itself.

Hence R is a field .

Option ( c )

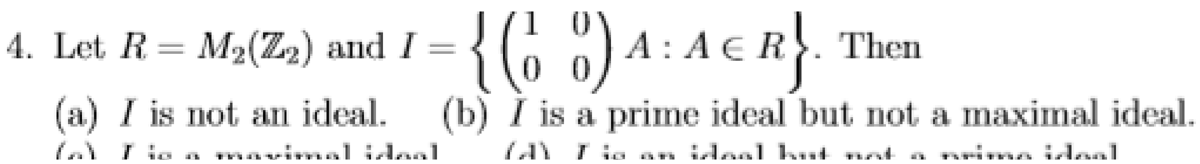


3. The number of maximal ideals in $\mathbb{Z}_{16}$ are
   (a) 1. (b) 2. (c) 3. (d) 4.

**Solution :-**

An ideal I in Zn is maximal if and only if I = ⟨p⟩ where p is a prime dividing n. In this case prime divisor of 16 are 1 , 2 . Hence there are two such ideals .

Option (b)



4. Let $R = M_2(\mathbb{Z}_2)$ and $I = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A : A \in R\right\}$. Then
   (a) $I$ is not an ideal. (b) $I$ is a prime ideal but not a maximal ideal.
   (c) $I$ is a maximal ideal. (d) $I$ is an ideal but not a prime ideal.

4. Let $R = M_2(\mathbb{Z}_2)$ and $I = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A : A \in R \right\}$. Then

(a) $I$ is not an ideal. (b) $I$ is a prime ideal but not a maximal ideal.

(c) $I$ is a maximal ideal. (d) $I$ is an ideal but not a prime ideal.

**Solution :-**

## $M_2(\mathbb{R})$ has no non-trivial two-sided ideals

Assuming $M_2(\mathbb{R})$ refers to the set of all $2 \times 2$ real matrices:

If $A$ is a rank 2 matrix, then $(A) = (I)$ is the whole ring. If $A$ is the zero matrix, then we get the trivial ideal. The only remaining possibility is that $A$ is a rank 1 matrix.

By performing row operations (i.e. multiplying on the left by units) and column operations (i.e. multiplying on the right by units), we can produce any other rank 1 matrix. So, we have $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in (A)$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in (A)$. Because $(A)$ is closed under addition, $I \in (A)$, which means that $(A)$ is the entire ring.

Thus, the only two two-sided ideals are the trivial ideal and the ring itself.

As for nontrivial one-sided ideals, consider

$$\left\{ A \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} : A \in M_2(\mathbb{R}) \right\}$$
$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} A : A \in M_2(\mathbb{R}) \right\}$$

Option (a)

5. Let $R = C[0,1]$, the ring of continuous real valued functions on [0,1] under pointwise addition and multiplication, $I = \{ f \in R : f(1/2) = 0 \}$.

(a) $I$ is not an ideal. (b) $I$ is a prime ideal but not a maximal ideal.

(c) $I$ is a maximal ideal. (d) $I$ is an ideal but not a prime ideal.

**Solution :-**

Let R be the ring of all continuous functions from [0,1] to $\mathbb{R}$ . Then all maximal ideals of S have the form $M_{x_0}$=$\{f \in S \mid f(x_0) = 0\}$

Option (c)

6. In the polynomial ring $\mathbb{Z}[x]$, consider $I = \{ f(x) : f(0) = 0 \}$, then

(a) $I$ is an ideal. (b) $I$ is prime ideal but not maximal ideal.

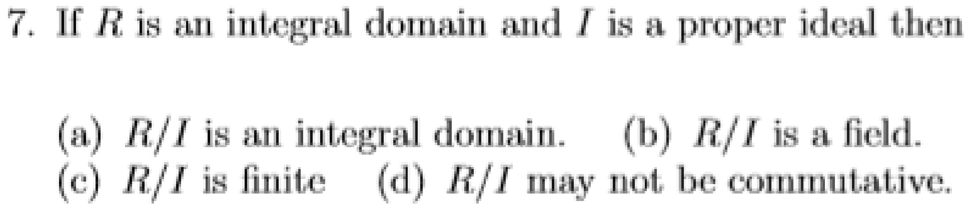
(c) $I$ is a maximal ideal. (d) $I$ is ideal but neither prime ideal nor maximal.

**Solution :-** Is a prime ideal of Z[x] but not an maximal ideal as

Nonconstant polynomials do not generate maximal ideals in $\mathbb{Z}[x]$

I = $\{ 0 + a_1 x + a_2 x^2 + \ ---- - a_n x^n \mid a_i \in \ \mathbb{R} \}$

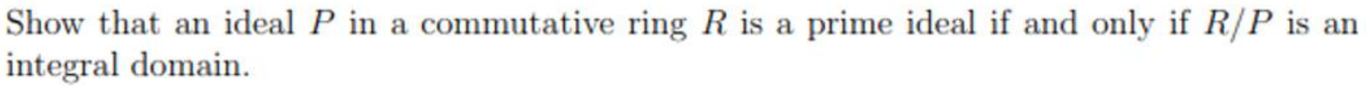
Option (b)



7. If $R$ is an integral domain and $I$ is a proper ideal then

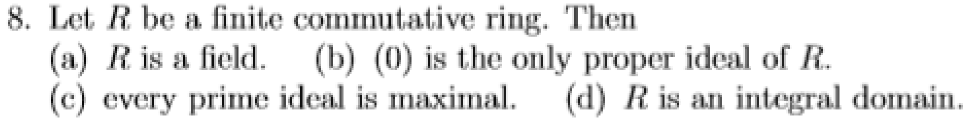
(a) $R/I$ is an integral domain. (b) $R/I$ is a field.
(c) $R/I$ is finite (d) $R/I$ may not be commutative.

**Solution :-**



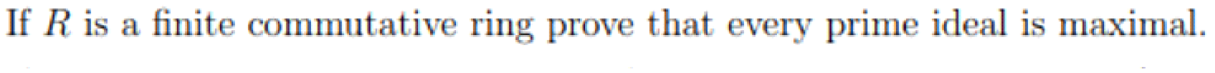
Show that an ideal $P$ in a commutative ring $R$ is a prime ideal if and only if $R/P$ is an integral domain.

Option (a)



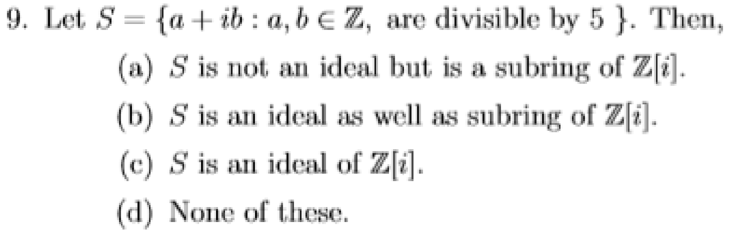
8. Let $R$ be a finite commutative ring. Then
(a) $R$ is a field. (b) (0) is the only proper ideal of $R$.
(c) every prime ideal is maximal. (d) $R$ is an integral domain.

**Solution :-** We know by theorem 7(a)



If $R$ is a finite commutative ring prove that every prime ideal is maximal.

Option (c)



9. Let $S = \{a + ib : a, b \in \mathbb{Z},$ are divisible by 5 $\}$. Then,
(a) $S$ is not an ideal but is a subring of $\mathbb{Z}[i]$.
(b) $S$ is an ideal as well as subring of $\mathbb{Z}[i]$.
(c) $S$ is an ideal of $\mathbb{Z}[i]$.
(d) None of these.

**Solution :- Option (b)**

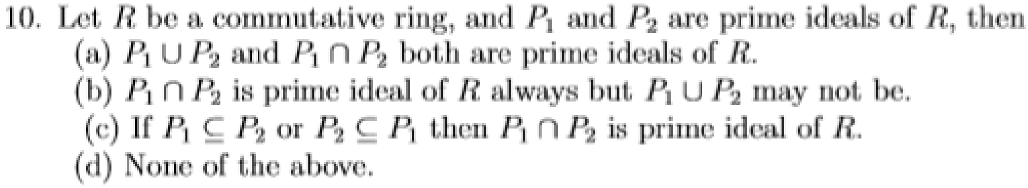
a+ib , c+id $\in S$ , $(a+ib)-(c+id) = (a-c)+i(b-d) \in S$

(a+ib )(c+id)$\in S$ S is a subring of Z[i]

Similarly , ( x+iy) $\in$ $Z[i]$

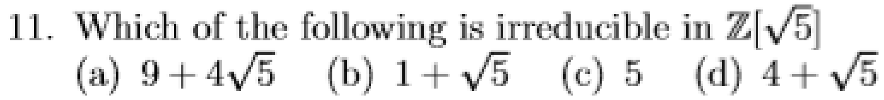
( x+iy)(a+ib ) , (a+ib )(x+iy) $\in S$

S is a also an ideal of Z[i]



10. Let $R$ be a commutative ring, and $P_1$ and $P_2$ are prime ideals of $R$, then
(a) $P_1 \cup P_2$ and $P_1 \cap P_2$ both are prime ideals of $R$.
(b) $P_1 \cap P_2$ is prime ideal of $R$ always but $P_1 \cup P_2$ may not be.
(c) If $P_1 \subseteq P_2$ or $P_2 \subseteq P_1$ then $P_1 \cap P_2$ is prime ideal of $R$.
(d) None of the above.

**Solution :- Option (b)**



11. Which of the following is irreducible in $\mathbb{Z}[\sqrt{5}]$
(a) $9+4\sqrt{5}$ (b) $1+\sqrt{5}$ (c) $5$ (d) $4+\sqrt{5}$

**Solution :- Option (d)**

$\mathbf{x} =$ a + b $\sqrt{\mathrm{d}}$ is irreducible if N(x) is prime where

N ( a + b $\sqrt{\mathrm{d}}$ ) $= \left| \mathrm{a}^2 - \mathrm{db}^2 \right|$

If , we take a + b $\sqrt{\mathrm{d}}$ $= 9 + 4\sqrt{5}$ , then

$\left| \mathrm{a}^2 - \mathrm{db}^2 \right| = |\, 9^2 - 5 \times 4^2 | = |\, 81 - 80 | = 1$

If N(x) = 1 then x is a unit ,

So , $9 + 4\sqrt{5}$ is a unit ,

If , we take a + b $\sqrt{\mathrm{d}}$ $=$ $1+\sqrt{5}$ ,

then , $\left| \mathrm{a}^2 - \mathrm{db}^2 \right| = |\, 1^2 - 5 \times 1^2 | = |\, 1 - 5 | = 4$ , which is not a prime.

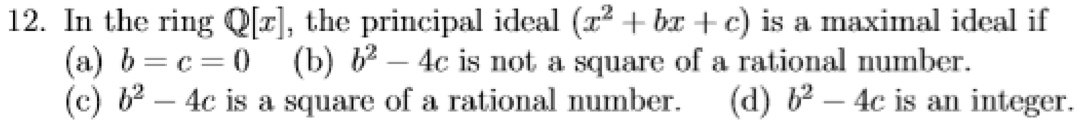
If , we take a + b $\sqrt{\mathrm{d}}$ $=$ $5$,

then , $\left| \mathrm{a}^2 - \mathrm{db}^2 \right| = |\, 5^2 - 0 | = |\, 25 | = 25$ , which is not a prime.

If , we take a + b $\sqrt{\mathrm{d}}$ $= 4 + \sqrt{5}$

then , $\left| \mathrm{a}^2 - \mathrm{db}^2 \right| = |\, 4^2 - 5 \times 1^2 | = |\, 16 - 5\, | = 11$ , which is not prime.

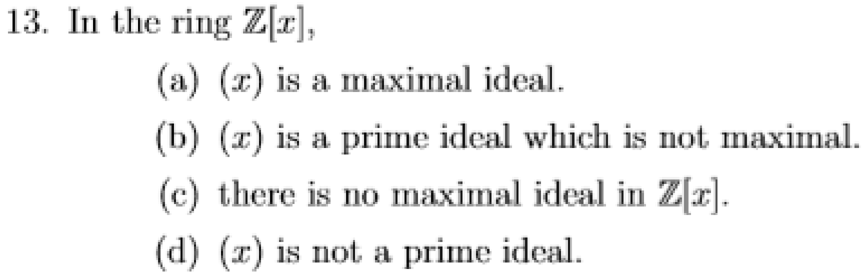
Hence , $4 + \sqrt{5}$ is irreducible in $\mathbb{Z}\left(\sqrt{5}\right)$



12. In the ring $\mathbb{Q}[x]$, the principal ideal $(x^2+bx+c)$ is a maximal ideal if
(a) $b=c=0$ (b) $b^2-4c$ is not a square of a rational number.
(c) $b^2-4c$ is a square of a rational number. (d) $b^2-4c$ is an integer.

**Solution :-** We know by theorem that $< x^2 + bx + c >$ is a maximal ideal of R[x] if $b^2 - 4c < 0$ , $a, b$ , $c \in R$

Option (b) ( $b^2 - 4c$) is not a square of a rational number



13. In the ring $\mathbb{Z}[x]$,

(a) $(x)$ is a maximal ideal.

(b) $(x)$ is a prime ideal which is not maximal.

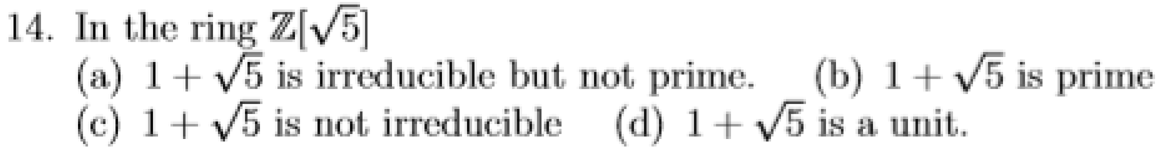
(c) there is no maximal ideal in $\mathbb{Z}[x]$.

(d) $(x)$ is not a prime ideal.

**Solution :-** $\mathbb{Z}[x]$ has maximal ideal $<$ f(x) $>$ if f(x) is irreducible in $\mathbb{Z}[x]$

As < x > is irreducible in $\mathbb{Z}[x]$ , $hence$

Option (a)



14. In the ring $\mathbb{Z}[\sqrt{5}]$

(a) $1+\sqrt{5}$ is irreducible but not prime. (b) $1+\sqrt{5}$ is prime

(c) $1+\sqrt{5}$ is not irreducible (d) $1+\sqrt{5}$ is a unit.

**Solution :-**

In $\mathbb{Z}\left(\sqrt{5}\right)$, N $(1+\sqrt{5}) = a^2 - 5b^2 = 1 - 5 = -4$

Which is not a unit $1+\sqrt{5}$ = xy in $\mathbb{Z}\left(\sqrt{5}\right)$

With neither x nor y a unit then as we have N(x) $= \pm 2$ which is not possible.

$\Rightarrow \left(1+\sqrt{5}\right)$ is irreducible .

$2.2 = 4 = \left(1+\sqrt{5}\right)\left(-1+\sqrt{5}\right)$

$\Rightarrow \left(1+\sqrt{5}\right) / 2.2$ ,

But for any a + b $\sqrt{5} \in \mathbb{Z}\left(\sqrt{5}\right)$

Hence we get , $(a+b\sqrt{5})\left(1+\sqrt{5}\right) = (a+5b) + (a+b)\sqrt{5}$

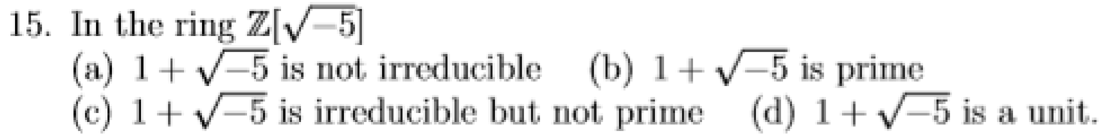
Which never equal to two since the system ,

$a + 5b = 2$ , $a + b = 0$ has no solution

$\Rightarrow \left(1+\sqrt{5}\right) \nmid 2$

$\Rightarrow \left(1+\sqrt{5}\right)$ is not a prime.

Option (a)



15. In the ring $\mathbb{Z}[\sqrt{-5}]$

(a) $1+\sqrt{-5}$ is not irreducible (b) $1+\sqrt{-5}$ is prime

(c) $1+\sqrt{-5}$ is irreducible but not prime (d) $1+\sqrt{-5}$ is a unit.

**Solution :-**

p is reducible in $\mathbb{Z}\left(\sqrt{-5}\right)$ iff p = $a^2 + 5b^2$ for some a , b in $\mathbb{Z}$ that iff

p = 20n + 1 or p = 20n + 9

(1) Any integer of the form $a^2 + 5b^2$ is reducible in $\mathbb{Z}\left(\sqrt{-5}\right)$ .

(2) If $a + b\sqrt{-5}$ is prime in $\mathbb{Z}\left(\sqrt{-5}\right)$ then its norm $a^2 + 5b^2$ is a prime in $\mathbb{Z}$ .

$1^2 + 5\left(\sqrt{-5}\right)^2 = 1 + 5\left(\sqrt{-5}\right)\left(\sqrt{-5}\right) = 1 + 5^2 = 26$

Which is not a prime.

Option (c) $1+\sqrt{-5}$ is irreducible but not a prime.

16. Consider the following pairs of elements in the given rings respectively. (i) $2+i$ and $1-2i$ in $\mathbb{Z}[i]$ (ii) $1-\sqrt{-5}$ and $7-3\sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$ (iii) 2 and $1+i$ in $\mathbb{Z}[i]$. Then
(a) (i) and (iii) are pairs of associates (b) (i) and (ii) are pairs of associates
(c) (i), (ii) and (iii) are pairs of associates. (d) only (iii) is a pair of associates.

**Solution :-**

**(i)** In a commutative ring a , b $\in R$ are said to be associates if b $=$ ua where u $\in R$ be an unit .

(2+i) = ( a + ib) (1-2i)

(2+i) = (a+2b)+i(b-2a)

Comparing both side we get , a + 2b = 2 ----------------(1)

-2a + b = 1 -----------------(2)

Then by solving (1) and (2) we get a = 0 , b = 1 ,

Therefore , ( 2 + i)= i(1-2i)

We know units of $\mathbb{Z}[i]$ is only $\pm 1$ , $\pm i$

Therefore , (2+i) and ( 1-2i) are associates.

**(ii)** 1 - $\sqrt{-5}$ = ( a + b $\sqrt{-5}$ ) $\left(7 - 3\sqrt{-5}\right)$

1 - $\sqrt{-5}$ = ( 7a + 15b ) + $\sqrt{-5}$ (-3a + 7b)

Comparing both side we get , 7a + 15b = 1 ----------------(3)

-3a + 7b = -1 -------------(4)

Solving (3) and (4) we get , b = $\frac{-4}{94} \notin \mathbb{Z}$

So , **(** 1 - $\sqrt{-5}$ ) and ( $7 - 3\sqrt{-5}$ ) are not associates .

(iii) 2 = ( a +ib) (1+i)= (a + b)i+(a-b)

Comparing both side we get a - b = 2 -----------------------------(5)

a + b = 0 ----------------------------(6)

Solving ( 5) and (6) we get , a= 1 , b = -1

So , 2 =(1+i)(1-i)

Hence , they are associates

Option (a)

17. Consider the following elements in $\mathbb{Z}[\sqrt{-5}]$ (i) $6+\sqrt{-5}$ (ii) 7 (iii) $2-3\sqrt{-5}$. Then
(a) (ii) and (iii) are irreducible and (i) is not irreducible
(b) (i) and (iii) are irreducible and (ii) is not irreducible
(c) (i),(ii) and (iii) are all irreducible (d) (i),(ii) and (iii) are all reducible.

**Solution : As**

$x = a + b\sqrt{d}$ is irreducible if N(x) is prime where

N ( a + b $\sqrt{d}$ ) $= \left| a^2 - db^2 \right|$

Take , $a + b\sqrt{d}$ = 6 + $\sqrt{-5}$ we get

N(6 + $\sqrt{-5}$ ) $= \left|6^2-(-5)1^2\right| = |36+5| = 41$ which is a prime .

So , 6 + $\sqrt{-5}$ is irreducible in $\mathbb{Z}\left(\sqrt{-5}\right)$

$a+b$ $\sqrt{d}$ = 7 = 7 + 0. $\sqrt{-5}$ we get

N(7 + 0. $\sqrt{-5}$) $= \left|7^2-(-5)0^2\right| = |49+0| = 49$ which is not a prime .

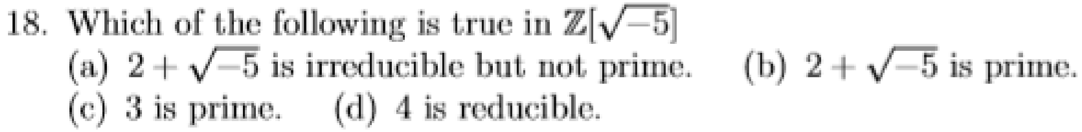
So , 7 is not irreducible in $\mathbb{Z}\left(\sqrt{-5}\right)$

Take , $a+b$ $\sqrt{d}$ = 2 – 3$\sqrt{-5}$ we get

N(2 – 3$\sqrt{-5}$ ) $= \left|2^2-(-5)(-3)^2\right| = |4+5\times 9| = 49$ which is not a prime .

So ,2 – 3$\sqrt{-5}$ is not irreducible in $\mathbb{Z}\left(\sqrt{-5}\right)$

**Therefore (ii) and (iii) is reducible , but (i) is not reducible.**



18. Which of the following is true in $\mathbb{Z}[\sqrt{-5}]$

(a) $2+\sqrt{-5}$ is irreducible but not prime. (b) $2+\sqrt{-5}$ is prime.

(c) 3 is prime. (d) 4 is reducible.

**Solution :-**

N( 2 +$\sqrt{-5}$ ) $= \left|2^2-(-5)1^2\right| = |4+5| = 9$

Which is not a prime . As

$x =$ $a+b$ $\sqrt{d}$ is irreducible if N(x) is prime where

N ( a + b $\sqrt{d}$ ) $= \left|a^2-db^2\right|$

So , 2 +$\sqrt{-5}$ is not irreducible $\mathbb{Z}\left(\sqrt{-5}\right) \Rightarrow it\ is\ not\ a\ prime.$

$N(3+0\sqrt{-5}$ ) = $|3^2-(-5)\times 0| = 9$ , which is not a prime .

So 3 is also not irreducible in $\mathbb{Z}\left(\sqrt{-5}\right)$

If we take , N(4) = 16 , which is not a prime .

Hence , 4 is reducible in $\mathbb{Z}\left(\sqrt{-5}\right)$

Option (d)

20. Which of the following is prime in $\mathbb{Z}[i]$,

(a) 2. (b) 5. (c) 17. (d) 3.

**Solution :-** Option (d)

2 = ( 1+i)(1-i) Hence 2 is reducible in $\mathbb{Z}[i] \Rightarrow 2\ is\ not\ a\ prime.$

$5 =$ (2+i)(2-i) , Hence 5 is reducible in $\mathbb{Z}[i] \Rightarrow\ 5\ is\ not\ a\ prime.$

17=( 4+i)(4-i) ,Hence 17 is reducible in $\mathbb{Z}[i] \Rightarrow\ 17\ is\ not\ a\ prime.$

But , 3 does not have any factor in $\mathbb{Z}[i]\ \Rightarrow 3\ is\ \ a\ prime.$

21. $a \pm ib$ is irreducible in $\mathbb{Z}[i]$ satisfying $a^2 + b^2 = p$ if

(a) $p$ is a prime integer. (b) $p$ is an odd integer.

(c) $p = 2$ or a prime such that $p \equiv 1 \bmod 4$. (d) None of these.

**Solution :-**

$a \pm ib$ is irreducible in $\mathbb{Z}[i]$ and sayisfing $a^2 + b^2 = p$ if p is a prime integer .

Option (a)

22. The quotient ring $\frac{\mathbb{Z}[i]}{(1+i)}$

(a) an integral domain which is not a field.

(b) a field having 2 elements.

(c) a field having 4 elements.

(d) a ring with proper zero divisors.

**Solution :- Option (b)**

Here , $\frac{\mathbb{Z}[i]}{1+i} = \{\, (1+i)(a+ib) | (a+ib) \in \mathbb{Z}[i]\}$

$= \left\{\, a + ib + ia + i^2 b \,\middle|\, (a+ib) \in \mathbb{Z}[i]\right\}$

$= \{\, (a-b) + i(a+b) |\, a, b \in \mathbb{Z}\,\}$

There , ( 1 + i)(1 - i) = 1+1 = 2

Therefore , $\frac{\mathbb{Z}[i]}{1+i}$ have all elements of $\mathbb{Z}_2 = \{\, 0 , 1\}$

23. The ring $\frac{\mathbb{R}[x]}{(x^4+1)}$ is

(a) an infinite integral domain. (b) an infinite field.

(c) a finite field. (d) None of these.

**Solution :-**

Here $x^4 + 1 = 0$

the solution of the above equation does not belong to $\mathbb{R}$.

Hence , the polynomial $x^4 + 1$ is irreducible in $\mathbb{R}[x]$.

Hence the polynomial $x^4 + 1$ is maximal ideal in $\mathbb{R}[x]$.

Therefore , $\frac{\mathbb{R}[x]}{x^4+1}$ $is\ a\ field$ which has infinite number of elements .

Hence , it is an infinite field.

Option (b).

24. Consider the ring homomorphisms $f_1 : \mathbb{Z}[i] \to \mathbb{Z}_2$ defined by $f(a+bi) = (a-b)\, mod\, 2$ and $f_2 : \mathbb{Z}[i] \to \mathbb{Z}_5$ defined by $f(a+bi) = (a-2b)\, mod\, 5$. Then

(a) ker $f_1$ is a maximal ideal but ker $f_2$ is not a maximal ideal.

(b) ker $f_2$ is a maximal ideal but ker $f_1$ is not a maximal ideal.

(c) both ker $f_1$ and ker $f_2$ are not maximal ideals.

(d) both ker $f_1$ and ker $f_2$ are maximal ideals.

**Solution :-**

Here ker $f_2$ is maximal ideal in < 2 + i >

Also , ker $f_1 = \{ a + ib \in \mathbb{Z}[i]$ , where f ( a+ib) = =0(mod2)}

={ $a + ib \in \mathbb{Z}[i]$ , where (a-b)(mod 2)=0(mod2)}

$\Rightarrow a - b \equiv 0 \ (mod\ 2)$

⇒ 2 / ( a- b)

$\Rightarrow a - b = 2k$ , for some k

$\Rightarrow a = 2k + b$

Hence , ker $f_1 = \{ a + ib \in \mathbb{Z}[i] \ , a = 2k + b \}$

= { (2k+b)+ib ∈ $\mathbb{Z}[i]$ , b ∈ $\mathbb{Z}$ }

= { 2k + ( 1+i )b , b ∈ $\mathbb{Z}$ }

= { ( 1+i)(1-i)k + ( 1+i )b , b ∈ $\mathbb{Z}$ }

= < 1+i> , which is also maximal ideal $\mathbb{Z}[i]$.

Option (d) , both $f_1, f_2$ are maximal ideals of $\mathbb{Z}[i]$.

25. In the ring $\mathbb{R}[x]$ and $\mathbb{C}[x]$, consider the ideal $I = (x^2 - x + 2)$

(a) $I$ is a maximal ideal in both $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

(b) $I$ is a maximal ideal in $\mathbb{R}[x]$ but not $\mathbb{C}[x]$.

(c) $I$ is a maximal ideal in $\mathbb{C}[x]$ but not in $\mathbb{R}[x]$.

(d) $I$ is a not maximal ideal in both $\mathbb{R}[x]$ and $\mathbb{C}[x]$.

**Solution :-**

Here , I = ( $x^2 - x + 2$) is irreducible in $\mathbb{R}[x]$.

But , I is reducible in $\mathbb{C}[x]$.

As , x $= \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 1 \times 2}}{2} = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm i\sqrt{7}}{2}$

So , x $= \frac{1+i\sqrt{7}}{2}, \frac{1-i\sqrt{7}}{2}$ are the roots of the polynomial **(** $x^2 - x + 2$).

Hence , **(** $x^2 - x + 2$) is redicible in $\mathbb{C}[x]$.

So , I is maximal in $\mathbb{R}[x]$ but not in $\mathbb{C}[x]$ .

Option (b)