LINEAR PROGRAMMING PROBLEM

Mathematical Formulation

Linear Programming Problem:

It is a technique for determining an optimum schedule of interdependent activities in view of the available resources. Programming is another word for planning and refers to the process of determining a particular plan of action from amongst several activities. The word linear stands for indicating that all relationships involved in a particular problem is linear.

Components of LPP:

A LPP consists of three components, namely the (i) decision variables (activities), (ii) the objective (goal) and (iii) the constraints(restrictions).

(i) The decision variables refer to the activities that are competing one another for sharing the resources available. These variables are usually inter-related in terms of utilisation of resources and need simultaneous solutions. All the decision variables are considered as continuous, controllable and non-negative.

(ii) A LPP must have an objective which should be clearly identifiable and measurable in quantitative terms. It could be of profit (sales) maximisation, cost (time) minimisation and so on. The relationship among the variables representing objective must be linear.

(iii) There are always certain limitations (or constraints) on the use of resources, such as labour, space, raw materials, money, etc. that limit the degree to which an objective can be achieved.

Such constraints must be expressed as linear inequalities or equalities in terms of decision variables.

Basic Assumptions:

The following four basic assumptions are necessary for all LPP:

(a) Certainty : In all LPP's , it is assumed that all the parameters , such as availability of resources, profit (or cost) contribution of a unit of decision variable and consumption of resources by a unit decision variable must be known and fixed. In other words, it means that all the coefficients in the objective function as well as in the constraints are completely known with certainty and do not change during the period of study.

(b) Divisibility (or continuity): This implies that solution values of the decision variables and resources can take on any non-negative values, including fractional values of the decision variables.

(c) Proportionality : This requires the contribution of each decision variable in both the objective function and the constraints to be directly proportional to the value of the variable.

(d) Additivity : The value of the objective function for the given values of decision variables and the total sum of resources used, must be equal to the sum of the contributions (profit or cost) earned from each decision variable and the sum of the resources used by each decision variable respectively.

MATHEMATICAL FORMULATION OF THE PROBLEM

The procedure for mathematical formulation of a linear programming problem consists of the following major steps :

Step 1: Study the given situation to find the key decision to be made.

Step 2 : Identify the variables involved and designate them by symbols x_j(j = 1, 2,…).

Step 3 : State the feasible alternatives which generally are : $x_j \geq 0$, for all j.

Step 4 : Identify the constraints in the problem and express them as linear inequalities or equations, LHS of which are linear functions of the decision variables.

Step 5 : Identify the objective function and express it as a linear function of the decision variables.

DEFINITIONS:

1. Concepts of solution : The solution is a set of values for each variable that:
   (i) are consistent with the constraints (i.e. feasible) &
   (ii) result in the best possible value of the objective function(i. e. optimal)
2. Feasible solution : A non-negative vector of variables that satisfies the constraints (restrictions) is called a feasible solution to the LPP. A feasible

solution that minimizes or maximizes the objective function is called an optimal solution. For example : the maximum profit or the minimum cost.

3. Basic solution: A basic solution is any solution of a LPP satisfying certain specific technical conditions, i.e., satisfies all the constraints (≤ , ≥ and = type constraints, i.e., all the inequality and equality constraints).
4. Basic feasible solution : A basic feasible solution is a solution which satisfies all the constraints and also non-negative restrictions.
5. Initial Basic feasible solution : The solution of Minimization in OR (also known as optimization) for our advantage in any scenario, let it be transportation, resources, cost etc. This is known as Initial Basic feasible solution.
6. Degenerate solution : A basic solution to the system is called degenerate if one or more of the basic variables vanish.
7. Associated cost vector: Let X_B be a basic feasible solution to the LPP:

   Maximize $z = cx$ subject to:

   $$Ax = b \text{ and } x \geq 0$$

   Then the vector $C_B = (C_{B1}, C_{B2},.......C_{Bn})$, where C_{Bi} are components of C associated with the basic variables, is called the cost vector associated with the basic feasible solution X_B.

   It is obvious that the value of the objective function for the basic feasible solution X_B is given by

   $$Z_0 = C_B X_B.$$

8. Improved basic feasible solution: Let X_B and $\hat{X}_B$ be two feasible solutions to the standard LPP. Then $\hat{X}_B$ is said to be an improved basic feasible solution , as compared to X_B, if

   $$\hat{C}_B\hat{X}_B \geq C_B X_B$$

   Where $\hat{C}_B$ is constituted of cost components corresponding to $\hat{X}_{B.}$
9. Optimum basic feasible solution : A basic feasible solution X_B to the LPP:

   Maximize $z = cx$ subject to : $Ax = b$ and $x \geq 0$ is called an optimum basic feasible solution if $Z_0 = C_B X_B \geq Z^*$ where Z^* is the value of objective function for any feasible solution.
10. Slack variable: It is a variable that is added to an inequality constraint to transform it into an equality.
11. Surplus variable : It is a variable that is subtracted fro an inequality constraint to transform it into an equality.

GRAPHICAL SOLUTION METHOD:

LPP involving two variables can easily be solved by graphical method. This method also provides an insight into the concept of Simplex Method- a powerful technique to solve the LPP involving three or more decision variables.

The major steps in the solution of a LPP by graphical method are summarised as follows:

Step 1 : Identify the problem – the decision variables , the objective and the restrictions.

Step 2 : Set up the mathematical formulation of the problem.

Step 3: Plot a graph representing all the constraints of the problem and identify the feasible region (solution space). The feasible region is the intersection of all the regions represented by the constraints of the problem and is restricted to the first quadrant only.

Step 4: The feasible region obtained in step 3 may be bounded or unbounded. Compute the co-ordinates of all the corner points of the feasible region.

Step 5 : Find out the value of the objective function at each corner (solution) point determined in Step 4.

Step 6 : Select the corner point that optimizes (maximizes or minimizes) the value of the objective function. It gives the optimum feasible solution.

General LPP:

Def: Let Z be a linear function on R^n defined by

(a) $Z = C_1x_1 + C_2x_2 + \ldots..+ C_nx_n$

Where C_j's are the constraints. Let (a_{ij})be an m*n real matrix and $\{b_1, b_2,\ldots\ldots b_m\}$ be a set of constraints such that

(b) '$a_{11}x_1 + a_{12}x_2+\ldots\ldots+a_{1n}x_n \geq$ or $\leq$ or $= b_1$

'$a_{21}x_1 + a_{22}x_2+\ldots\ldots+a_{2n}x_n \geq$ or $\leq$ or $= b_2$

…………………………………………………….

'$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots.+ a_{mn}x_n \geq$ or $\leq$ or $= b_m$

And finally let

(c) $x_j \geq 0$, j = 1, 2, ……n.

The problem of determining an n-tuple $(x_1, x_2, \ldots x_n)$ which makes Z a minimum or maximum and satisfies (b) and (c) is called general LPP.

Canonical Form:

The general formulation of LPP discussed earlier can always be put in the following form:

Maximize $Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n$

Subject to the constraints

'$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \leq b$, $i=1, 2, 3, \ldots, m$

And $x_1, x_2, \ldots, x_n \geq 0$

By making use of some elementary transformation . This is called the Canonical Form of LPP.

Standard Form:

The General LPP in the form

Maximize or minimize

$Z = C_1x_1 + C_2x_2 + \ldots + C_nx_n$ subject to the constraints:

$a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n = b_i$, $i= 1, 2, \ldots m$

$x_1, x_2, \ldots, x_n \geq 0$

is known as the Standard Form.

SIMPLEX METHOD:

Basic Solution: Given a system of m simultaneous linear equations in n known (m < n)

$Ax + b, \quad x^T \in R^n$

Where A = m * n matrix of rank m.

Let B be any sub-matrix formed by m linearly independent columns of A.

Computational Procedure:

The two fundamental conditions on which the simplex method is based on are :

(i) Condition of feasibility: It assumes that if the initial (starting) solution is basic feasible , then during computation only basic feasible solution will be obtained.

(ii) Condition of optimality: It guarantees that only better solutions will be encountered.

			C_1	C_2	…	…	….	…	…	…	C_n
C_B	y_B	X_B	‘y_1	‘y_2	….	…	……	…	…	…	‘y_n
C_{B1}	‘y_{B1}	X_{B1}	‘y_{11}	‘y_{12}	…	…	…	…	…	…	‘y_{1n}
C_{B2}	‘y_{B2}	X_{B2}	‘y_{21}	‘y_{22}	…	…	…	…	…	…	‘y_{2n}
…	…	…									
C_{Bn}	‘y_{Bn}	X_{Bn}	‘y_{m1}	‘y_{m2}	…	…	…	…	…	…	‘y_{mn}
		‘z_0	‘z_1-C_1	‘z_2-C_2							‘z_n-C_n

Where X_B = basic feasible solution

C_B = associated cost vector

Y_{ij}’s = column vector corresponding to basic as well as non-basic variables.

Use of Artificial Variables:

Use of slack variables is very convenient as the starting basic variable. Thus if the original constraint is an equation or is of type ≥ we may no longer have a ready starting basic feasible solution.

In order to obtain an initial basic feasible solution, we first put the given LPP into its standard form and then a non-negative variable is added to the left side of each of the equations that lacks the much needed starting basic variables. The so-added variable is called **Artificial Variable** and plays the same role as a slack variable in providing the initial basic feasible solution. But we must take them out in the final level. It is used in:

(I) Two-Phase Method

(II) Big-M Method or Method of Penalties.

Big-M Method:

The Big-M Method is an alternative method of solving a LPP involving artificial variables. In this method we assign a very high penalty (say M) to the artificial variables in the objective function.

The iterative procedure of the algorithm is given below:

Step 1 : Write the given LPP into its standard form and check whether there exists a starting basic feasible solution.

(a) If there is ready starting basic feasible solution , move to Step 3.
(b) If there does not exist a ready starting basic feasible solution , move to Step 2.

Step 2: Add artificial variables to the left side of each equation that has no obvious starting basic variables. Assign a very high penalty (M) to these variables in the objective function.

Step 3: Apply Simplex Method to the modified LPP. Following cases may arise at the last iteration:

(a) At least one artificial variable is present in the basis with zero value. In such a case the current optimum basic feasible solution is degenerate.
(b) At least one artificial variable is present in the basis with a positive value. In such a case, the given LPP does not possess an optimum basic feasible solution. The given problem is said to have a pseudo-optimum basic feasible solution.

DUALITY:

Introduction: Associated with every LPP (maximization or minimization) there always exists another LPP which is based on the same data and having the same solution. The original problem is called the **Primal Problem** while the associated one is called the **Dual Problem.** We can treat either of them as primal and the other dual. The two problems thus constitute **primal-dual** pair.

Standard Primal Problem:

1. Max $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

   Subject to the constraints:

   $a_{i1}x_1 + a_{i2}x_2 + \ldots\ldots\ldots + a_{in}x_n = b_i$; $i = 1, 2, \ldots\ldots, m$; $x_j \geq o$, $j=1, 2, \ldots., n$

   Dual : Min $z^* = b_1w_1 + b_2w_2 + \ldots\ldots + b_mw_m$

   Subject to the constraints :

   $a_{1j}w_1 + a_{2j}w_2 + \ldots\ldots + a_{mj}w_m \geq c_j$, $j = 1, 2, \ldots.., n$; $w_i(i=1,2,\ldots,m)$unrestricted.

   Here, x_j's are primal variables , w_i's are dual variables and the constants are as usual.

Standard Primal Objective		Dual	
	Objective	Constraints	Variables
Maximization	Minimization	≥	Unrestricted
Minimization	Maximization	≤	Unrestricted

Importance of Duality:

Duality in LPP is essentially a unifying theory that develops the relationship between a given linear problem and another related linear problem stated in terms of variables with this shadow price interpretation. The following are the importance of Duality:

1. Yields powerful theorem.
2. Computation is easy.
3. Solution of the dual checks the accuracy if there is any error in the primal solution.
4. Gives additional information as to how the optimum solution changes.
5. Economic interpretation of the dual help the management in making future decisions.

Characteristics or Properties of Dual:

1. Dual of the dual is Primal.
2. If either the primal or the dual has a solution, then there also has a shadow solution. The optimum value of both the solution will be same.
3. If any of the primal or dual is infeasible, then the other has an unbounded (infinite) solution.

Economic Interpretation of the dual LP:

. Suppose an entrepreneur wants to purchase all of Fulkerson's resources (plumber, finishing, carpentry).

. What prices should he/she offer for the resources that will entice Fulkerson to sell.

- Optimal dual solution <=> ''fair'' prices for associated resources:

-known as ‘ marginal prices ’ or ‘shadow prices’.

Strong Duality Theorem:

1. If the primal LP has finite optimal value, then:
- The dual has finite optimal value and
- The primal and dual have the same optimal value.

2. If the primal and dual have feasible solutions, then
- Both LPs have finite optimal values, and
- The primal and dual have the same optimal value.

Weak Duality Theorem:
For any Primal-Dual pair of LPs, the objective function value of any feasible solution to the maximising LP is always less than equal to the objective function value of any feasible solution to the minimising LP.

Corollary 1 : If the primal and dual have feasible solution value, then these solutions must be optimal for the primal and the dual.
Corollary 2 : For any primal-dual pair of LPs, if one of the LPs is unbounded , then the other must be infeasible.
Note: The reverse doesn’t always hold good i.e., if one of the LPs is infeasible , the other is not necessarily be unbounded.

INTEGER PROGRAMMING:

A LPP in which all or some of the decision variables are constrained to assume non-negative integer values is called **Integer Programming Problem.** This type of problem is of particular importance in business and industry, where quite often, the fractional solutions are unrealistic because the units are not divisible.

The LPP with the additional requirements that the variables can take on only integer values may have the following mathematical form

Maximize or Minimize $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

Subject to the constraints:

$$a_{i1}x_1 + a_{i2}x_2 + \dots\dots + a_{in}x_n = b_i\ ; \quad i=1, 2, \dots\dots, m; j=1, 2, \dots, n$$

and $x_j \geq 0$,

where x_j are integer valued for $j = 1, 2, \dots, p$ $(p \leq n)$.

PURE & MIXED IPP:

An IPP in which all variables are required to be integers is called a pure or all-integer programming problem.

An IPP in which only some of the variables are required to be integers, is known as a mixed integer programming problem.

An IPP in which all the variables must have integer values only zero or unity, is called the zero-one integer programming problem.

GOMORY'S ALL I.P.P. METHOD:

A systematic procedure for obtaining an optimum integer solution to an all-integer programming problem was first suggested by R.E.Gomory. His method starts without taking into consideration the integer requirements. If the solution so obtained is integral then the current solution is optimum . However, if some of the basic variables are not integer valued then an additional linear constraint called Gomory's constraint (fractional cut) is generated. After having generated a linear constraint (cutting plane), it is added as the last row of the optimum simplex table indicating that the solution is no longer feasible. The modified problem is then solved by using dual simplex method. An optimum integer solution is obtained if all the variables in the solution are integer valued, otherwise another Gomory constraint is added and the procedure is repeated. The optimum integer solution will be reached eventually after necessary new constraints have been added to drive away all the superior non-integer solutions.