

3-04-2021

Practical - 04

Practical no 4. Homomorphism, Isomorphism of Rings

1. Consider the ring $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.

Let $I = \{(a, -a) : a \in \mathbb{Z}\}$, $J = \{(a, 0) : a \in \mathbb{Z}\}$. Then,

- (a) I and J are ideals of $\mathbb{Z} \times \mathbb{Z}$.
- (b) I is an ideal of $\mathbb{Z} \times \mathbb{Z}$ but J is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.
- (c) Neither I nor J is an ideal of $\mathbb{Z} \times \mathbb{Z}$.
- (d) J is an ideal of $\mathbb{Z} \times \mathbb{Z}$ but I is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.

Solution :-

Consider $I = \{(a, -a) : a \in \mathbb{Z}\}$ then

Take, $(a, -a), (b, -b) \in I$, so

$$(a, -a) - (b, -b) = (a - b, -(a - b)) \in I \text{ as } (a - b) \in \mathbb{Z}$$

Again, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we get

$$(x, y) \times (a, -a) = (xa, -ya) \notin I$$

I is not an ideal of $\mathbb{Z} \times \mathbb{Z}$

Let us consider $J = \{(a, 0) : a \in \mathbb{Z}\}$ Then,

Take, $(a, 0), (b, 0) \in J$ so

$$(a, 0) - (b, 0) = (a - b, 0) \in J \text{ as } (a - b) \in \mathbb{Z}$$

Again, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we get

$$(x, y) \times (a, 0) = (xa, 0) \in J \text{ as } xa \in \mathbb{Z}$$

$$(a, 0) \times (x, y) = (ax, 0) \in J \text{ as } ax \in \mathbb{Z}$$

J is an ideal of $\mathbb{Z} \times \mathbb{Z}$.

So we get I is not an ideal but J is an ideal of $\mathbb{Z} \times \mathbb{Z}$.

Option - (d)

2. Consider the ring $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ and

let $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5 \right\}$. Then

- (a) I is a subring of $M_2(\mathbb{Z})$ but not an ideal.
- (b) I is an ideal of $M_2(\mathbb{Z})$ but not a subring.
- (c) I is not an ideal of $M_2(\mathbb{Z})$.
- (d) I is both a subring and an ideal of $M_2(\mathbb{Z})$.

Solution :- Here, $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ and

$$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5 \right\}$$

$$\text{Let, } \begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix}, \begin{pmatrix} 5l & 5m \\ 5n & 5p \end{pmatrix} \in I$$

$$\text{Then, } \begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} - \begin{pmatrix} 5l & 5m \\ 5n & 5p \end{pmatrix} = \begin{pmatrix} 5x - 5l & 5y - 5m \\ 5z - 5n & 5w - 5p \end{pmatrix} \in I$$

As, $(5x - 5l), (5y - 5m), (5z - 5n), (5w - 5p)$ is divisible by 5

Now take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ for, $a, b, c, d \in \mathbb{Z}$

$$\text{And, } \begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 5ax + 5yc & 5xb + 5yd \\ 5za + 5wc & 5zb + 5wd \end{pmatrix} \in I$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} \in I$$

Hence I is an ideal of $M_2(\mathbb{Z})$.

Now let's check if I is a subring.

$$\begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix}, \begin{pmatrix} 5l & 5m \\ 5n & 5p \end{pmatrix} \in I$$

$$\begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} \times \begin{pmatrix} 5l & 5m \\ 5n & 5p \end{pmatrix} = \begin{pmatrix} 25xl + 25yn & 25xm + 25yp \\ 25zl + 25wn & 25zm + 25wp \end{pmatrix} \in I$$

So I is also a subring of $M_2(\mathbb{Z})$.

Hence I is an ideal as well as subring.

Option - (d)

3. Consider the ideal $I = 10\mathbb{Z}$ and $J = 12\mathbb{Z}$, then

(a) $I + J = 22\mathbb{Z}, IJ = 120\mathbb{Z}$. (b) $I + J = 2\mathbb{Z}, IJ = 60\mathbb{Z}$.

(c) $I + J = 2\mathbb{Z}, IJ = 120\mathbb{Z}$. (d) None of these.

Solution :-

$$I = 10\mathbb{Z} = \{ \dots, -30, -20, -10, 0, 10, 20, 30, \dots \}$$

$$J = 12\mathbb{Z} = \{ \dots, -36, -24, -12, 0, 12, 24, 36, \dots \}$$

Hence, $I + J = 2\mathbb{Z}$, $IJ = 120\mathbb{Z}$

Option (c)

4. In the ring of integers \mathbb{Z} , consider the ideals $I = 4\mathbb{Z} + 6\mathbb{Z}, J = m\mathbb{Z} + n\mathbb{Z}, m, n \in \mathbb{N}$. Then,

(a) $I = 24\mathbb{Z}, J = mn\mathbb{Z}$ (b) $I = 2\mathbb{Z}, J = d\mathbb{Z}$ where $d = \gcd(m, n)$

(c) $I = 12\mathbb{Z}, J = l\mathbb{Z}$, where $l = \text{lcm}(m, n)$ (d) None of the above.

Solution :-

In the ring of integers \mathbb{Z}

$$I = 4\mathbb{Z} + 6\mathbb{Z} \text{ and } J = m\mathbb{Z} + n\mathbb{Z}, m, n \in \mathbb{N}$$

$$I = 2\mathbb{Z}, J = d\mathbb{Z} \text{ where } d = \gcd(m, n)$$

Option (b)

5. In the ring of integers \mathbb{Z} , consider the ideal $I = (6\mathbb{Z})(4\mathbb{Z})$, then

(a) $I = 24\mathbb{Z}$. (b) $I = 12\mathbb{Z}$. (c) $I = 2\mathbb{Z}$. (d) None of these.

Solution :- $I = (6\mathbb{Z}) \cap (4\mathbb{Z}) = 2\mathbb{Z}$

Option (c)

6. $(n\mathbb{Z})(m\mathbb{Z}) = (n\mathbb{Z}) \cap (m\mathbb{Z})$ if and only if
 (a) $m|n$. (b) $(m, n) = 1$. (c) $m = n$. (d) None of these.

Solution :- $(n\mathbb{Z}) \cap (m\mathbb{Z}) = (n\mathbb{Z})(m\mathbb{Z})$ only when $(m, n) = 1$
 Option (b)

7. Which of the following is true

- (i) \mathbb{Z} is an ideal of \mathbb{Q} (ii) $\{(n, n) : n \in \mathbb{Z}\}$ is an ideal of $\mathbb{Z} \times \mathbb{Z}$
 (iii) $\{f \in F(\mathbb{R}, \mathbb{R}) : f(\pi) = 0\}$ where $F(\mathbb{R}, \mathbb{R})$ is a ring of real valued functions.
 (a) (i), (ii) (b) (ii), (iii) (c) only (iii). (d) None of these.

Solution :- \mathbb{Z} is not an ideal of \mathbb{Q} as for any $\frac{p}{q} \in \mathbb{Q}$ and $a \in \mathbb{Z}$ we get

$\frac{p}{q} \times a = \frac{pa}{q} \notin \mathbb{Z}$ as \mathbb{Z} is set of integers.

$I = \{(n, n) : n \in \mathbb{Z}\}$ is not an ideal of $\mathbb{Z} \times \mathbb{Z}$.

As, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ and $(n, n) \in I$ then

$(x, y) \times (n, n) = (xn, yn) \notin I$

$\{f \in F(\mathbb{R}, \mathbb{R}) : f(\pi) = 0\}$ where $F(\mathbb{R}, \mathbb{R})$ is an ideal

Option (c)

8. The number of ring homomorphisms from \mathbb{Q} to itself is

- (a) 1 (b) 2 (c) infinitely many (d) none of these.

Solution :- The number of homomorphism from \mathbb{Q} to \mathbb{Q}

If $f : \mathbb{Q} \rightarrow \mathbb{Q}$ is a homomorphism, prove that $f(x) = 0$ for all $x \in \mathbb{Q}$ or $f(x) = x$ for all x in \mathbb{Q} .

Hence, there are two homomorphism from \mathbb{Q} to \mathbb{Q}

Option (b)

9. The number of ring homomorphisms from \mathbb{C} to itself is

- (a) 1 (b) 2 (c) infinitely many (d) none of these.

Solution :-

Only such ring homomorphism are identity and conjugate

i.e. $f(a + ib) = a + ib$ and $f(a + ib) = a - ib$.

Then there are two such homomorphism.

Option (b)

10. Consider the following pair of rings.

- (i) $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$ (ii) $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-5}]$ (iii) \mathbb{Q} and \mathbb{R} (iv) $M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\}, \mathbb{C}$

(a) (i) and (iv) are isomorphic pairs of rings.

(b) (i) and (ii) are isomorphic pairs of rings.

(c) only (iv) is an isomorphic pair of rings.

(d) (i), (ii) and (iv) are isomorphic pairs of rings.

Solution :- $\mathbb{Z}(\sqrt{2})$ and $\mathbb{Z}(\sqrt{5})$ are not isomorphic.

$$\mathbb{Z}(\sqrt{2}) = \{ a + \sqrt{2}b, a, b \in \mathbb{Z} \} \text{ and}$$

$$\mathbb{Z}(\sqrt{5}) = \{ a + \sqrt{5}b, a, b \in \mathbb{Z} \}$$

Suppose $\mathbb{Z}(\sqrt{2})$ and $\mathbb{Z}(\sqrt{5})$ are isomorphic then

$$f(1) = 1$$

Then for $x \in \mathbb{Z}, x = (1 + 1 + 1 \dots + 1)x$ times

$$f(x) = f(1) + f(1) + f(1) + \dots + f(1) \text{ x times}$$

$$= 1 + 1 + 1 + \dots + 1$$

$$= x \quad \forall x \in \mathbb{Z}$$

Since, f is a bijection, $f(\sqrt{2}) = \sqrt{5}$

$$\text{Now, } 2 = f(2) = f(\sqrt{2} \cdot \sqrt{2}) = f(\sqrt{2}) \cdot f(\sqrt{2})$$

$$2 = \sqrt{5} \times \sqrt{5} = 5$$

Which is not possible

Therefore, $\mathbb{Z}(\sqrt{2})$ and $\mathbb{Z}(\sqrt{5})$ are not isomorphic.

Hence (i) is not true.

$$f : \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \rightarrow a + ib, \text{ given by } f \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + ib$$

Option :- (c)

Remaining Objectives :-



11. Consider the following maps from $M_2(\mathbb{Z}_p) \rightarrow \mathbb{Z}_p$ defined by

$$f \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a, \quad g \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d, \quad h(A) = \det A \text{ for } A \in M_2(\mathbb{Z}_p).$$

(a) f, g, h are all ring homomorphisms. (b) only h is a ring homomorphism.

(c) f is a ring homomorphism and g, h are not. (d) none of these.

$$\text{Solution :- } f \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \text{ and } f \begin{pmatrix} x & y \\ z & w \end{pmatrix} = x$$

$$\text{Let us take, } \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(\mathbb{Z}_p)$$

$$f \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix} = a+x = f \begin{pmatrix} a & b \\ c & d \end{pmatrix} + f \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$f \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = f \begin{pmatrix} ax+bz & bw+ay \\ cx+dz & cy+dw \end{pmatrix} = ax+bz \neq ax$$

$$\neq f \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times f \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

So f is not a homomorphism.

$$\text{Consider, } g \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a+d \text{ and } g \begin{pmatrix} x & y \\ z & w \end{pmatrix} = x+w$$

$$\begin{aligned} \text{Then, } g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) &= g\left(\begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix}\right) = (a+x) + (d+w) \\ &= (a+d) + (x+w) \\ &= g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) + g\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) \end{aligned}$$

$$\begin{aligned} \text{Again, } g\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x & y \\ z & w \end{pmatrix}\right) &= f\left(\begin{pmatrix} ax+bz & bw+ay \\ cx+dz & cy+dw \end{pmatrix}\right) \\ &= (ax+bz) + (cy+dw) \\ &\neq (a+d)(x+w) \end{aligned}$$

Hence g is not a homomorphism.

$$h(A) = \det A$$

$$\text{So, } h(A+B) = \det(A+B) = \det A + \det B = h(A) + h(B)$$

$$\text{Again, } h(AB) = \det(AB) = \det A \times \det B = h(A) \cdot h(B)$$

Hence g is not a homomorphism.

Option (b)

12. Consider the map $\pi_1 : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ and $\pi_2 : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ defined by $\pi_1(m, n) = m$, $\pi_2(m) = (m, 0)$, where $\mathbb{Z} \times \mathbb{Z}$ denotes ring with component wise addition and multiplication.

- (a) π_1 and π_2 are ring homomorphisms.
- (b) Both π_1, π_2 are not ring homomorphisms.
- (c) π_1 is a ring homomorphism but π_2 is not a ring homomorphism.
- (d) π_2 is a ring homomorphism but π_1 is not a ring homomorphism.

Solution :- (b)

$$\pi_1 : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}, \pi_1(m, n) = m$$

$$\text{Then, } \pi_1((m, n) + (a, b))$$

$$= \pi_1(m+a, n+b)$$

$$= m+a$$

$$= \pi_1(m, n) + \pi_1(a, b)$$

$$\text{Again, } \pi_1((m, n) \times (a, b)) = \pi_1(ma, nb) = ma = \pi_1(m, n)\pi_1(a, b)$$

Hence, π_1 is a group homomorphism.

Consider, $\pi_2 : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$ we get

$$\pi_2(m) = (m, 0) \text{ and } \pi_2(x) = (x, 0)$$

$$\text{Then, } \pi_2(m+x) = (m+x, 0) = (m, 0) + (x, 0) = \pi_2(m) + \pi_2(x)$$

Hence, π_2 is a group homomorphism.

So both π_1 and π_2 are ring homomorphism.

13. The number of ring homomorphisms from \mathbb{Z} to \mathbb{Z} are

- (a) one
- (b) zero
- (c) two
- (d) infinitely many.

Solution :- (c)

Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a ring homomorphism. Note that for $n \in \mathbb{Z}$, $f(n) = nf(1)$. Thus f is completely determined by its value on 1. Since 1 is an idempotent in \mathbb{Z} (i.e. $1^2 = 1$), then $f(1)$ is again idempotent. Now we need to determine all of the idempotents of \mathbb{Z} . To this end, take $x \in \mathbb{Z}$ such that $x^2 = x$. Thus $x^2 - x = x(x - 1) = 0$. Since \mathbb{Z} is an integral domain, we deduce that either $x = 0$ or $x = 1$. Thus the complete list of idempotents of \mathbb{Z} are 0 and 1. Thus $f(1)$ being idempotent implies that either $f(1) = 0$ or $f(1) = 1$. In the first case, $f(n) = 0$ for all n and in the second case $f(n) = n$ for all n . Thus, the only ring homomorphisms from \mathbb{Z} to \mathbb{Z} are the zero map and the identity map.

14. Let $\phi_n : \mathbb{Z}[x] \rightarrow \mathbb{Z}_n[x]$ be defined by $\phi(a_0 + a_1x + \dots + a_kx^k) = \bar{a}_0 + \bar{a}_1x + \dots + \bar{a}_kx^k$, is a ring homomorphism only if
- (a) n is a prime number. (b) n is a positive integer.
(c) n is an odd integer. (d) n is an even integer.

Solution : -

$$\begin{aligned} \phi(a_0 + a_1x + \dots + a_kx^k) &= a_0 + a_1x + \dots + a_kx^k \\ \phi((a_0 + a_1x + \dots + a_kx^k) + (b_0 + b_1x + \dots + b_kx^k)) & \\ = (a_0 + b_0) + (a_1 + b_1)x + \dots + (a_k + b_k)x^k & \\ = \phi(a_0 + a_1x + \dots + a_kx^k) + \phi(b_0 + b_1x + \dots + b_kx^k) & \\ \phi((a_0 + a_1x + \dots + a_kx^k) \times (b_0 + b_1x + \dots + b_kx^k)) & \\ = \phi(a_0b_0 + (a_0b_1 + b_0a_1)x + \dots + a_kb_kx^{2k}) & \\ = a_0b_0 + (a_0b_1 + b_0a_1)x + \dots + a_kb_kx^{2k} & \end{aligned}$$

Option (b)

15. The kernel of the ring homomorphism $\phi : \mathbb{R}[x] \rightarrow \mathbb{C}$ defined by $\phi(f(x)) = f(2 + i)$ is
- (a) $\{f(x) \in \mathbb{R}[x] : (x - 2) | f(x)\}$. (b) $\{f(x) \in \mathbb{R}[x] : (x^2 - 4x - 5) | f(x)\}$.
(c) $\{f(x) \in \mathbb{R}[x] : (x^2 - 4x + 2) | f(x)\}$. (d) $\{f(x) \in \mathbb{R}[x] : (x^2 - 4x + 5) | f(x)\}$

Solution : -

Here, $\phi(f(x)) = f(2 + i)$

The kernel of ϕ is

$$\ker \phi = \{f(x) \in \mathbb{R}[x] : \phi(f(x)) = 0 + i0\}$$

$$\Rightarrow f(2 + i) = 0$$

So, $x = (2 + i)$ is a solution of $f(x) = 0$

$$\text{Hence, } (x - 2)^2 = i^2$$

$$x^2 - 4x + 4 = -1 \Rightarrow (x^2 - 4x + 5) = 0$$

So, $(x^2 - 4x + 5)$ is a factor of $f(x)$.

So, $(x^2 - 4x + 5) | f(x)$, Option - (d)

16. Consider the ring homomorphism $\phi : \mathbb{R}[x] \rightarrow \mathbb{R}$ defined by $\phi(a_0 + a_1x + \dots + a_nx^n) = a_0 + a_1 + \dots + a_n$. Then, the kernel ϕ is
- (a) $\{f(x) \in \mathbb{R}[x] : f(1) = 1\}$. (b) $\{f(x) \in \mathbb{R}[x] : f(1) = 0\}$.
(c) $\{f(x) \in \mathbb{R}[x] : f(0) = 1\}$. (d) None of these.

Solution : -

Here let us consider

$$\phi(a_0 + a_1x + \dots + a_nx^n) = a_0 + a_1 + \dots + a_n$$

Let us find the kernel of ϕ .

$\ker\phi$

$$= \{f(x) = (a_0 + a_1x + \dots + a_nx^n) \in \mathbb{R}[x] \mid \phi(a_0 + a_1x + \dots + a_nx^n) = 0\}$$

$$= \{f(x) = (a_0 + a_1x + \dots + a_nx^n) \in \mathbb{R}[x] \mid a_0 + a_1 + \dots + a_n = 0\}$$

$$= \{(a_0 + a_1x + \dots + a_nx^n) \in \mathbb{R}[x] \mid f(1) = 0\}$$

Option (b)

17. Consider the ring homomorphism $\phi : \mathbb{R}[x] \rightarrow \mathbb{R}$ defined by $\phi(a_0 + a_1x + \dots + a_nx^n) =$

$$\sum_{k=0}^n (-1)^k a_k. \text{ Then, the kernel } \phi \text{ is}$$

(a) $\{f(x) \in \mathbb{R}[x] : f(-1) = 0\}$. (b) $\{f(x) \in \mathbb{R}[x] : f(-1) = 1\}$.

(c) $\{f(x) \in \mathbb{R}[x] : f(-1) = -1\}$. (d) None of these.

Solution : –

Here let us consider the given map as

$$\phi(a_0 + a_1x + \dots + a_nx^n) = \sum_{k=0}^n (-1)^k a_k$$

Therefore the kernel ϕ

$$= \{(a_0 + a_1x + \dots + a_nx^n) \in \mathbb{R}[x] \mid \phi(a_0 + a_1x + \dots + a_nx^n) = 0\}$$

$$= \{f(x) = (a_0 + a_1x + \dots + a_nx^n) \in \mathbb{R}[x] \mid \sum_{k=0}^n (-1)^k a_k = 0\}$$

$$= \{f(x) = (a_0 + a_1x + \dots + a_nx^n) \in \mathbb{R}[x] \mid f(-1) = 0\}$$

Option (a)

18. Ring $H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ is

(a) is not isomorphic to $\mathbb{Z}[\sqrt{2}]$.

(b) is isomorphic to $\mathbb{Z}[\sqrt{2}]$.

(c) is isomorphic to $\mathbb{Q}[\sqrt{2}]$.

(d) None of these.

Solution : – (b)

Here, $H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}$ then it will be isomorphic to $\mathbb{Z}(\sqrt{2})$ where the function can be defined as

$$f: \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \rightarrow \mathbb{Z}(\sqrt{2}), \quad f\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\right) = a + b\sqrt{2} \text{ where } a, b \in \mathbb{Z}$$

19. Consider the maps $\phi_1 : M_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ defined by $\phi_1\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = a$ and $\phi_2 : R \rightarrow \mathbb{Z}$ defined

by $\phi_1\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\right) = a$, where $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z} \right\}$. Then

(a) ϕ_1 is a ring homomorphism but ϕ_2 is not a ring homomorphism.

(b) Both ϕ_1, ϕ_2 are ring homomorphisms.

19. Consider the maps $\phi_1 : M_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ defined by $\phi_1 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a$ and $\phi_2 : R \rightarrow \mathbb{Z}$ defined

by $\phi_2 \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = a$, where $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z} \right\}$. Then

- (a) ϕ_1 is a ring homomorphism but ϕ_2 is not a ring homomorphism.
- (b) Both ϕ_1, ϕ_2 are ring homomorphisms.
- (c) ϕ_2 is a ring homomorphism but ϕ_1 is not a ring homomorphism.
- (d) Both ϕ_1, ϕ_2 are not ring homomorphisms.

Solution : – $\phi_1 : M_2(\mathbb{Z}) \rightarrow \mathbb{Z}$ defined as

$$\phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a \quad \text{and} \quad \phi_1 \begin{pmatrix} x & y \\ z & w \end{pmatrix} = x$$

Let us take, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(\mathbb{Z}_p)$

$$\phi_1 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix} = a+x = \phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \phi_1 \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$\phi_1 \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} x & y \\ z & w \end{pmatrix} \right) = \phi_1 \begin{pmatrix} ax+bz & by+aw \\ cx+dz & cy+dw \end{pmatrix} = ax+bz \neq ax$$

$$\neq \phi_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \phi_1 \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

So ϕ_1 is not a homomorphism.

$\phi_2 : R \rightarrow \mathbb{Z}$ defined as where $R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{Z} \right\}$

$$\phi_2 \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} = a \quad \text{and} \quad \phi_2 \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} = x$$

Let us take, $\begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \in R$

$$\phi_2 \left(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \right) = \begin{pmatrix} a+x & b+y \\ c+z & 0 \end{pmatrix} = a+x = \phi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \phi_2 \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$\phi_2 \left(\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \times \begin{pmatrix} x & y \\ z & 0 \end{pmatrix} \right) = \phi_2 \begin{pmatrix} ax+bz & ay \\ cx & cy \end{pmatrix} = ax+bz \neq ax$$

$$\neq \phi_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \phi_2 \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

So ϕ_2 is not a homomorphism.

Option (d)

20. The question ring $\frac{\mathbb{Z}[i]}{(1+i)}$ is

- (a) is an infinite ring. (b) a field having 2 elements.
- (c) a ring having 4 elements. (d) a ring with proper zero-divisors.

Solution : – (b)

The norm of $a + bi$ is defined to be $(a + bi)(a - bi)$. It is also equal to $|\mathbf{Z}[i]/(a + bi)|$. This is part of a larger theory in Algebraic Number Theory. Let us see what happens in this specific example.

The elements of $\mathbf{Z}[i]/(1 + i)$ are the cosets of $(1 + i)$ and they are respectively $(1 + i)$ and $1 + (1 + i)$. Why is this? Take an element $a + bi \in \mathbf{Z}[i]$. By the division theorem, we can write $a + bi = q(1 + i) + r$ with $N(r) < N(1 + i) = 2$. That is $r = 0, 1$ or i . Hence the cosets are $0 + (1 + i), 1 + (1 + i)$ and $i + (1 + i)$.

Notice also that $1 + (1 + i) = i + (1 + i)$ since $1 - i = -i(1 + i) \in (1 + i)$.

Therefore $n = 2$ and you can show that $\mathbf{Z}[i]/(1 + i) \cong \mathbf{Z}/2\mathbf{Z}$.

Practical 4 Descriptive Question

1. Check whether following sets are ideals of the ring $\mathbb{Z} \times \mathbb{Z}$ under component wise addition and multiplication.

- (a) $I = \{(a, a) : a \in \mathbb{Z}\}$. (b) $I = \{(2a, 2b) : a, b \in \mathbb{Z}\}$.
 (c) $I = \{(2a, 0) : a \in \mathbb{Z}\}$. (d) $I = \{(a, -a) : a \in \mathbb{Z}\}$.

Solution :-

Consider $I = \{(a, a) : a \in \mathbb{Z}\}$ then

Take $(a, a), (b, b) \in I$, so

$$(a, a) - (b, b) = (a - b, a - b) \in I \text{ as } (a - b) \in \mathbb{Z}$$

Again, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we get

$$(x, y) \times (a, a) = (xa, ya) \notin I$$

I is not an ideal of $\mathbb{Z} \times \mathbb{Z}$

Consider $I = \{(2a, 2b) : a, b \in \mathbb{Z}\}$ then

Take $(2a, 2b), (2c, 2d) \in I$, as $a, b, c, d \in \mathbb{Z}$, so

$$(2a, 2b) - (2c, 2d) = (2a - 2c, 2b - 2d) \in I \text{ as } (2a - 2c), (2b - 2d) \in \mathbb{Z}$$

Again, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we get

$$(x, y) \times (2a, 2b) = (2xa, 2yb) \in I$$

$$(2a, 2b) \times (x, y) = (2ax, 2by) \in I$$

I is an ideal of $\mathbb{Z} \times \mathbb{Z}$

Let us consider $I = \{(2a, 0) : a \in \mathbb{Z}\}$ Then,

Take $(2a, 0), (2b, 0) \in I$ so

$$(2a, 0) - (2b, 0) = (2a - 2b, 0) \in I \text{ as } (a - b) \in \mathbb{Z}$$

Again, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we get

$$(x, y) \times (2a, 0) = (2xa, 0) \in I \text{ as } xa \in \mathbb{Z}$$

$$(2a, 0) \times (x, y) = (2ax, 0) \in I$$

So I is an ideal of $\mathbb{Z} \times \mathbb{Z}$.

Consider $I = \{ (a, -a) : a \in \mathbb{Z} \}$ then

Take $(a, -a), (b, -b) \in I$, so

$(a, -a) - (b, -b) = (a - b, -(a - b)) \in I$ as $(a - b) \in \mathbb{Z}$

Again, $(x, y) \in \mathbb{Z} \times \mathbb{Z}$ we get

$(x, y) \times (a, -a) = (xa, -ya) \notin I$

I is not an ideal of $\mathbb{Z} \times \mathbb{Z}$

2. Check which of the following are ideals of the polynomial ring $\mathbb{Z}[x]$.

(a) $I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : 3|a_0 \}$.

(b) $I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : 3|a_2 \}$.

(c) $I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : f(0) = 0 \}$.

(d) $I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : \sum_{i=0}^n a_i = 0 \}$.

Solution :-

(a) $I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : 3|a_0 \}$

Let, $f(x) = a_0 + a_1x + \dots + a_nx^n$ and

$g(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$ where $3|a_0, 3|b_0$

Then, $f(x) - g(x)$

$$= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \dots + (a_n - b_n)x^n \in I$$

As, $3|a_0, 3|b_0 \Rightarrow 3|(a_0 - b_0)$

Hence, $f(x) - g(x) \in I$

Again, Let $c \in \mathbb{R}$ we get

$$cf(x) = c(a_0 + a_1x + \dots + a_nx^n) \\ = ca_0 + ca_1x + \dots + ca_nx^n \in I$$

Similarly we can show $f(x)c \in I$

As $3|a_0$ then $3|ca_0, 3|a_0c$, for $c \in \mathbb{R}$

Therefore, I is an ideal of $\mathbb{Z}[x]$.

(b) $I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : 3|a_2 \}$

Let, $f(x) = a_0 + a_1x + \dots + a_nx^n$ and

$g(x) = b_0 + b_1x + b_2x^2 + \dots + b_nx^n$ where $3|a_2, 3|b_2$

Then, $f(x) - g(x)$

$$= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \dots + (a_n - b_n)x^n \in I$$

As, $3|a_2, 3|b_2 \Rightarrow 3|(a_2 - b_2)$

Hence, $f(x) - g(x) \in I$

Again, Let $c \in \mathbb{R}$ we get

$$cf(x) = c(a_0 + a_1x + \dots + a_nx^n) \\ = ca_0 + ca_1x + ca_2x^2 + \dots + ca_nx^n \in I$$

Similarly we can show $f(x)c \in I$

As $3|a_2$ then $3|ca_2, 3|a_2c$

Therefore, I is an ideal of $\mathbb{Z}[x]$.

$$(c) I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : f(0) = 0 \}$$

$$= \{ 0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] \}$$

$$\text{Let, } f(x) = 0 + a_1x + \dots + a_nx^n,$$

$$g(x) = 0 + b_1x + \dots + b_nx^n$$

$$f(x), g(x) \in I$$

$$\text{Then, } f(x) - g(x)$$

$$= (0 - 0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \dots + (a_n - b_n)x^n \in I$$

$$\text{Hence, } f(x) - g(x) \in I$$

Again, Let $c \in \mathbb{R}$ we get

$$cf(x) = c(0 + a_1x + \dots + a_nx^n)$$

$$= 0 + ca_1x + ca_2x^2 + \dots + ca_nx^n \in I$$

$$f(x)c = 0 + a_1cx + a_2cx^2 + \dots + a_ncx^n \in I$$

Therefore, I is an ideal of $\mathbb{Z}[x]$.

$$(d) I = \{ f(x) = a_0 + a_1x + \dots + a_nx^n \in \mathbb{Z}[x] : \sum_{i=1}^n a_i = 0 \}$$

$$\text{Let, } f(x) = a_0 + a_1x + \dots + a_nx^n \text{ then}$$

$$a_0 + a_1 + a_2 + \dots + a_n = 0$$

$$g(x) = b_0 + b_1x + \dots + b_nx^n \text{ then}$$

$$b_0 + b_1 + b_2 + \dots + b_n = 0$$

$$\text{Hence, } f(x), g(x) \in I$$

$$\text{Then, } f(x) - g(x)$$

$$= (a_0 - b_0) + (a_1 - b_1)x + (a_2 - b_2)x^2 + \dots + (a_n - b_n)x^n \in I$$

$$\text{As, } a_0 + a_1 + a_2 + \dots + a_n = 0 \quad \text{---(1)}$$

$$b_0 + b_1 + b_2 + \dots + b_n = 0 \quad \text{---(2)}$$

$$\text{By (1) - (2)}$$

$$\text{Then, } (a_0 - b_0) + (a_1 - b_1) + (a_2 - b_2) + \dots + (a_n - b_n) = 0$$

Again, Let $c \in \mathbb{R}$ we get

$$cf(x) = c(a_0 + a_1x + \dots + a_nx^n)$$

$$= ca_0 + ca_1x + ca_2x^2 + \dots + ca_nx^n \in I$$

$$\text{As, } a_0 + a_1 + a_2 + \dots + a_n = 0 \text{ then}$$

$$ca_0 + ca_1 + ca_2 + \dots + ca_n = 0$$

$$\text{Similarly we can show } f(x)c \in I$$

Therefore, I is an ideal of $\mathbb{Z}[x]$.

3. (a) Let R be a commutative ring and $a \in R$ be non-zero. Show that, annihilator of a , $\text{ann}(a) = \{r \in R : ra = 0\}$ is an ideal of R .
- (b) If A, B are ideals of a commutative ring R such that $R = A + B$, show that $A \cap B = AB$.
- (c) If A, B are ideals of a commutative ring R such that $A \cap B = (0)$ then show that $ab = 0$ for $a \in A, b \in B$.

Solution :-

(a) Let R be a commutative ring and $a \in R$ be non zero .

Annihilator $\text{ann}(a) = \{ r \in R \mid ra = 0 \}$

Let $r, s \in \text{ann}(a)$

$$\Rightarrow ra = 0, sa = 0$$

$$\Rightarrow ra - sa = 0$$

$$\Rightarrow (r - s)a = 0$$

$$\Rightarrow r - s \in \text{ann}(a)$$

Let $x \in R$ and $r \in \text{ann}(a)$, $ra = 0$

$$\Rightarrow x(ra) = 0$$

$$\Rightarrow (xr)a = 0 \Rightarrow xr \in \text{ann}(a)$$

As R is a commutative ring Similarly we can show that $rx \in \text{ann}(a)$

Hence $\text{ann}(a)$ is a ideal of R .

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(b) A and B are ideals of a commutative ring R such that $R = A + B$

To show that $A \cap B = AB$.

Solution: First, suppose that $x \in AB$. Then $x = a_1b_1 + a_2b_2 + \dots + a_nb_n$ for some $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$. Note that, for any $i \in \{1, 2, \dots, n\}$, we have that $a_ib_i \in A$ since $a_i \in A$, $b_i \in B$, and A is an ideal of R . Similarly, $a_ib_i \in B$ since $b_i \in B$, $a_i \in R$, and B is an ideal of R . Hence, $a_1b_1 + a_2b_2 + \dots + a_nb_n \in A \cap B$. Thus, $AB \subseteq A \cap B$ (note that this is always true - we did not use the assumption that $A + B = R$ here).

Now, suppose that $y \in A \cap B$. Since $1 \in R = A + B$, there exist $a \in A$ and $b \in B$ such that $1 = a + b$. Then $y = y(1) = y(a + b) = ya + yb = ay + yb$. Since $y \in B$, $ay \in AB$. Since $y \in A$, $yb \in AB$. Hence, $y = ay + yb \in AB$. Thus, $A \cap B \subseteq AB$. Hence, $A \cap B = AB$.

(c) **Solution :-** As A and B are ideals of a commutative Ring R with

$$A \cap B = (0)$$

$$\text{Let } x \in A \cap B = (0) \Rightarrow x = 0$$

Now we know that in previous sum we got, $A \cap B = AB$

So, $a \in A$ and $b \in B$ we get $ab \in AB = A \cap B = (0)$

Hence, $ab = 0$.

4. Let $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even}\}$. Show that S is a subring of $\mathbb{Z}[i]$ but not an ideal of $\mathbb{Z}[i]$.

Solution :- $S = \{a + ib : a, b \in \mathbb{Z}, b \text{ is even}\}$

To show that S is a subring of $\mathbb{Z}[i]$

Let, $a + ib, c + id \in \mathbb{Z}[i]$ where b and d is even.

So, $(a + ib) - (c + id) = (a - c) + i(b - d) \in S$ as b and d is even so $(b - d)$ is also even.

Again, $(a + ib)(c + id) = (ac - bd) + i(ad + bc) \in S$

as b and d is even so $(ad + bc)$ is also even.

Then, S is a subring of $\mathbb{Z}[i]$.

5. Show that $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are even integers} \right\}$ is an ideal of $M_2(\mathbb{Z})$.

Solution :-

Here, $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ and

$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 2 \right\}$

Let, $\begin{pmatrix} 2x & 2y \\ 2z & 2w \end{pmatrix}, \begin{pmatrix} 2l & 2m \\ 2n & 2p \end{pmatrix} \in I$

Then, $\begin{pmatrix} 2x & 2y \\ 2z & 2w \end{pmatrix} - \begin{pmatrix} 2l & 2m \\ 2n & 2p \end{pmatrix} = \begin{pmatrix} 2x - 2l & 2y - 2m \\ 2z - 2n & 2w - 2p \end{pmatrix} \in I$

As, $(2x - 2l), (2y - 2m), (2z - 2n), (2w - 2p)$ are also even

Now take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ for, $a, b, c, d \in \mathbb{Z}$

And, $\begin{pmatrix} 2x & 2y \\ 2z & 2w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2xa + 2yc & 2xb + 2yd \\ 2za + 2wc & 2zb + 2wd \end{pmatrix} \in I$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 2x & 2y \\ 2z & 2w \end{pmatrix} = \begin{pmatrix} 2ax + 2cy & 2bx + 2dy \\ 2az + 2cw & 2bz + 2dw \end{pmatrix} \in I$

Hence I is an ideal of $M_2(\mathbb{Z})$.

6. Show that $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ is an ideal of the ring $I = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$.

Solution :-

Here, $I = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R} \right\}$ and

$I = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$

Let, $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in I$

Then, $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a - b \\ 0 & 0 \end{pmatrix} \in I$

As, $a - b \in \mathbb{R}$

Now take $\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \in I$ for $a_1, b_1, c_1 \in \mathbb{R}$

And, $\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a_1 a \\ 0 & 0 \end{pmatrix} \in I$ as $a_1 a \in \mathbb{R}$

$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} = \begin{pmatrix} 0 & a a_1 \\ 0 & 0 \end{pmatrix} \in I$ as $a a_1 \in \mathbb{R}$

Hence I is an ideal of I .

7. Show that $I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5 \right\}$ is an ideal of $M_2(\mathbb{Z})$.

Solution :-

Here, $M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \right\}$ and

$I = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \text{ are divisible by } 5 \right\}$

Let, $\begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix}, \begin{pmatrix} 5l & 5m \\ 5n & 5p \end{pmatrix} \in I$

Then, $\begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} - \begin{pmatrix} 5l & 5m \\ 5n & 5p \end{pmatrix} = \begin{pmatrix} 5x - 5l & 5y - 5m \\ 5z - 5n & 5w - 5p \end{pmatrix} \in I$

As, $(5x - 5l), (5y - 5m), (5z - 5n), (5w - 5p)$ is divisible by 5

Now take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ for, $a, b, c, d \in \mathbb{Z}$

And, $\begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 5xa + 5yc & 5xb + 5yd \\ 5za + 5wc & 5zb + 5wd \end{pmatrix} \in I$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 5x & 5y \\ 5z & 5w \end{pmatrix} = \begin{pmatrix} 5ax + 5cy & 5bx + 5dy \\ 5az + 5cw & 5bz + 5dw \end{pmatrix} \in I$

Hence I is an ideal of $M_2(\mathbb{Z})$.

8. Is $I = \{4a + bi : a, b \in \mathbb{Z}\}$ an ideal of $\mathbb{Z}[i]$? Justify.

Solution :-

$I = \{4a + ib : a, b \in \mathbb{Z}\}$

To show that I is a subring of $\mathbb{Z}[i]$

Let, $4a + ib, 4c + id \in \mathbb{Z}[i]$ where $a, b, c, d \in \mathbb{Z}$

So, $(4a + ib) - (4c + id) = (4a - 4c) + i(b - d) = 4(a - c) + i(b - d) \in I$

$(a - c), (b - d) \in \mathbb{Z}$

Let, $(x + iy) \in \mathbb{Z}[i]$

Again, $(4a + ib)(x + iy) = (4ax - by) + i(4ay + bx) \notin I$ here

Similarly we can show that $(x + iy)(4a + ib) \notin I$

Then, I is not an ideal of $\mathbb{Z}[i]$.

9. Let I be an ideal of ring R. Show that, $I^m = \left\{ \sum_{i=1}^n a_{i1} a_{i2} \cdots a_{im} : a_{ij} \in R, n \in \mathbb{N} \right\}$ is an ideal of R.

Solution :- $I^m = \left\{ \sum_{i=1}^n a_{i1} \cdot a_{i2} \cdots a_{im} \in R : n \in \mathbb{N} \right\}$

Let us take, $\sum_{i=1}^n a_{i1} \cdot a_{i2} \cdots a_{im}, \sum_{i=1}^p b_{i1} \cdot b_{i2} \cdots b_{im} \in I^m$

Then, $\sum_{i=1}^n a_{i1} \cdot a_{i2} \cdots a_{im} - \sum_{i=1}^p b_{i1} \cdot b_{i2} \cdots b_{im}$

$= (a_{11} a_{12} a_{13} \cdots a_{1m}) + (a_{21} a_{22} a_{23} \cdots a_{2m}) + \cdots + (a_{n1} a_{n2} a_{n3} \cdots a_{nm}) - (b_{11} b_{12} b_{13} \cdots b_{1m}) - (b_{21} b_{22} a_{23} \cdots b_{2m}) - \cdots - (b_{p1} b_{p2} a_{p3} \cdots b_{pm}) \in I^m$

Again, let $c \in \mathbb{R}$

So, $c \times \sum_{i=1}^n a_{i1} \cdot a_{i2} \dots \dots \dots a_{im}$

$$= c [(a_{11}a_{12}a_{13} \dots \dots a_{1m}) + (a_{21}a_{22}a_{23} \dots \dots a_{2m}) + \dots \dots \dots + (a_{n1}a_{n2}a_{n3} \dots \dots a_{nm})]$$

$$= (c \cdot a_{11}a_{12}a_{13} \dots \dots a_{1m}) + (c \cdot a_{21}a_{22}a_{23} \dots \dots a_{2m}) + \dots \dots \dots + (c \cdot a_{n1}a_{n2}a_{n3} \dots \dots a_{nm}) \in I^m$$

Similarly, $\sum_{i=1}^n a_{i1} \cdot a_{i2} \dots \dots \dots a_{im} \times c \in I^m$

Hence, I^m is an ideal of \mathbb{R} .

10. Find the characteristic of $\mathbb{Z}[i] / \langle 2 + i \rangle$.

Solution :- $\phi : \mathbb{Z}[i] \rightarrow \mathbb{Z}_5$ is defined as

$$\phi(a + ib) = (a - 2b) \pmod{5}$$

(i) To show that ϕ is well define

Suppose, $a + ib = c + id$

$$\Rightarrow a = c \text{ and } b = d$$

$$\Rightarrow a - 2b = c - 2d$$

$$\Rightarrow \phi(a + ib) = \phi(c + id)$$

ϕ is well define.

(ii) To show that ϕ is onto.

$$\phi(0 + i0) = 0 - 2 \times 0 = 0 \pmod{5}$$

$$\phi(1 + i0) = 1 - 2 \times 0 = 1 \pmod{5}$$

$$\phi(2 + i0) = 2 - 2 \times 0 = 2 \pmod{5}$$

$$\phi(3 + i0) = 3 - 2 \times 0 = 3 \pmod{5}$$

$$\phi(4 + i0) = 4 - 2 \times 0 = 4 \pmod{5}$$

Therefore, every element of \mathbb{Z}_5 has a pre-image in $\mathbb{Z}[i]$.

Therefore, ϕ is onto.

(ii) To show that ϕ is ring homomorphism.

$$\phi((a + ib) + (c + id))$$

$$= \phi((a + c) + i(b + d))$$

$$= \overline{(a + c) - 2(b + d)}$$

$$= \overline{(a - 2b)} + \overline{(c - 2d)}$$

$$= \phi((a + ib)) + \phi((c + id))$$

$$\phi((a + ib) \times (c + id)) = \phi[(ac - bd) + i(ad + bc)]$$

$$= \overline{(a - 2d)(c - 2d)}$$

$$= \overline{(a - 2b)} \times \overline{(c - 2d)}$$

$$= \phi((a + ib)) \times \phi((c + id))$$

Hence, ϕ is a ring homomorphism.

$$\begin{aligned}
\text{Ker } \phi &= \{ a + ib \mid \phi(a + ib) = \bar{0} \} \\
&= \{ a + ib \mid (a - 2b) = \bar{0} \pmod{5} \} \\
&= \{ a + ib \mid a - 2b = 5k, k \in \mathbb{Z} \} \\
&= \{ (5k + 2b) + ib \mid b \in \mathbb{R}, k \in \mathbb{Z} \} \\
&= \{ b(2 + i) + (2 + i)(2 - i)k \mid b \in \mathbb{R}, k \in \mathbb{Z} \} \\
&= \{ (2 + i)(b + (2 - i)k) \mid b \in \mathbb{R}, k \in \mathbb{Z} \} \\
&= \langle 2 + i \rangle
\end{aligned}$$

$\phi : \mathbb{Z}[i] \rightarrow \mathbb{Z}_5$ is a ring onto homomorphism with
 $\text{ker } \phi = \langle 2 + i \rangle$

Therefore by first isomorphism theorem

$$\mathbb{Z}[i] / \langle 2 + i \rangle \cong \mathbb{Z}_5.$$

As \mathbb{Z}_5 is a field $\mathbb{Z}[i] / \langle 2 + i \rangle$ is also a field and $\text{char } \mathbb{Z}_5 = 5$.

Hence, $\text{char}(\mathbb{Z}[i] / \langle 2 + i \rangle) = 5$

11. Let R be a ring with prime characteristic p and $f : R \rightarrow R$ is defined by $f(a) = a^p$ for $a \in R$. Show that f is a ring homomorphism (called the Frobenius homomorphism). Is it an isomorphism?

Solution :- R is a ring with prime characteristic p .

$f : R \rightarrow R$ defined by $f(a) = a^p$ for $a \in R$.

(Called the Frobenius Homomorphism)

$$f(a + b) = (a + b)^p$$

$$\begin{aligned}
&= a^p + \binom{p}{1} a^{p-1}b + \binom{p}{2} a^{p-2}b^2 + \dots + \binom{p}{p-1} ab^{p-1} + b^p \\
&= a^p + p a^{p-1}b + \frac{p(p-1)}{2} a^{p-2}b^2 + \dots + pab^{p-1} + b^p
\end{aligned}$$

As the $\text{char } R = p$ we get

$$f(a + b) = a^p + 0 + 0 + \dots + 0 + b^p = f(a) + f(b)$$

$$\text{Again, } f(ab) = (ab)^p$$

$$\text{As } p \text{ is prime, } f(ab) = (ab)^p = a^p \cdot b^p = f(a) \cdot f(b)$$

Therefore, f is a ring homomorphism.

Let, $a = b$

$$\Rightarrow a^p = b^p$$

$$\Rightarrow f(a) = f(b)$$

Hence, f is well defined.

Conversely, let us suppose $f(a) = f(b)$

$$\Rightarrow a^p = b^p$$

$$\nRightarrow a = b$$

Therefore, f is not 1 - 1

Therefore, f is not isomorphic.

12. Show that the following are isomorphic:

$$(a) H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\} \text{ and } \mathbb{Z}[\sqrt{2}].$$

$$(b) M = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \text{ and } \mathbb{C}.$$

Solution :- (a) Let us define a function $f: \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \rightarrow \mathbb{Z}\sqrt{2}$ as

$$f\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\right) = a + b\sqrt{2}$$

(i) To show that f is 1-1.

$$\text{Let } f\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\right) = f\left(\begin{pmatrix} c & 2d \\ d & c \end{pmatrix}\right)$$

$$\Rightarrow a + b\sqrt{2} = c + d\sqrt{2}$$

Comparing both side we get , $a = c$ and $b = d$

$$\text{So we get , } \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}$$

Hence , f is 1-1.

(ii) To show that f is onto.

Let us take , $y = a + b\sqrt{2} \in \mathbb{Z}(\sqrt{2})$ then there exists $x = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ such that

$$f(x) = f\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\right) = a + b\sqrt{2} = y$$

Hence , for every $y = a + b\sqrt{2}$ here exists $x = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ such that $f(x) = y$

So , f is onto.

(iii) To show that f is ring homomorphism.

$$f\left(\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}\right)\right) = f\left(\begin{pmatrix} a+c & 2b+2d \\ b+d & a+c \end{pmatrix}\right)$$

$$= (a+c) + (b+d)\sqrt{2}$$

$$= (a+b\sqrt{2}) + (b+d\sqrt{2})$$

$$= f\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\right) + f\left(\begin{pmatrix} c & 2d \\ d & c \end{pmatrix}\right)$$

$$f\left(\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \times \begin{pmatrix} c & 2d \\ d & c \end{pmatrix}\right)\right) = f\left(\begin{pmatrix} ac+2bd & 2ad+2bc \\ bc+ad & 2bd+ac \end{pmatrix}\right)$$

$$= (ac+2bd) + (ad+bc)\sqrt{2}$$

$$= (a+b\sqrt{2})(c+d\sqrt{2})$$

$$= f\left(\begin{pmatrix} a & 2b \\ b & a \end{pmatrix}\right) \times f\left(\begin{pmatrix} c & 2d \\ d & c \end{pmatrix}\right)$$

Therefore by (i) , (ii) , (iii) we get f is a ring homomorphism.

(b) Similarly in this case take the function as

$f: \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \rightarrow \mathbb{C}$ as $f\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = a + ib$ and show that it is one-one, onto and ring homomorphism.

13. Let R be a ring and $a \in R$ be a unit. Show that the map from R into itself given by $x \mapsto axa^{-1}$ is a ring automorphism.

Solution :- Here R be a ring and $a \in R$ be a unit .

$f: R \rightarrow R$ defined as $f(x) = axa^{-1}$

To show that f is a ring automorphism .

(1) Let us show f is 1 - 1 .

Take $f(x) = f(y)$, where $x, y \in R$

$$\Rightarrow axa^{-1} = aya^{-1}$$

Operating both side by a^{-1} and a we get

$$a^{-1}(axa^{-1})a = a^{-1}(aya^{-1})a$$

$$(a^{-1}a)x(a^{-1}a) = (a^{-1}a)y(a^{-1}a)$$

$$exe = eye$$

$$x = y$$

Therefore f is one one .

(2) To show that f is onto.

Let us take $y = axa^{-1}$ then there exists $x = a^{-1}ya$ such that

$$f(x) = f(a^{-1}ya) = a(a^{-1}ya)a^{-1}$$

$$= (aa^{-1})y(aa^{-1})$$

$$= eye$$

$$f(x) = y$$

Therefore f is onto.

(3) To show that f is a ring homomorphism

$$f(x + y)$$

$$= a(x + y)a^{-1}$$

$$= axa^{-1} + aya^{-1}$$

$$= f(x) + f(y)$$

$$f(x \cdot y) = a(x \cdot y)a^{-1} = ax(a^{-1}a)ya^{-1}$$

$$= (axa^{-1})(aya^{-1})$$

$$= f(x) \cdot f(y)$$

Therefore by (1), (2), (3) we get f is a ring automorphism.

14. Suppose I and J are ideals of ring R so that $R = I + J$. Let $\phi: R \rightarrow R/I \times R/J$ be given by $\phi(r) = (r + I, r + J)$, then show that ϕ is a ring homomorphism with $\ker \phi = I \cap J$. Hence or otherwise show that the rings $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ when $(n, m) = 1$.

Solution :- Here $\phi(r) = (r + I, r + J)$

Then, $\phi(r_1 + r_2) = (r_1 + r_2 + I, r_1 + r_2 + J)$

$$\begin{aligned}
&= \left((r_1 + I) + (r_2 + I), (r_1 + J) + (r_2 + J) \right) \\
&= (r_1 + I, r_1 + J) + (r_2 + I, r_2 + J) \\
&= \phi(r_1) + \phi(r_2) \\
&\text{Again, } \phi(r_1 r_2) \\
&= (r_1 r_2 + I, r_1 r_2 + J) \\
&= \left((r_1 + I)(r_2 + I), (r_1 + J)(r_2 + J) \right) \\
&= (r_1 + I, r_1 + J)(r_2 + I, r_2 + J) \\
&= \phi(r_1) \times \phi(r_2)
\end{aligned}$$

Therefore, ϕ is a ring homomorphism.

$$\begin{aligned}
\text{Ker } \phi &= \{ r \in R \text{ then } \phi(r) = (r + I, r + J) = (0 + I, 0 + J) \} \\
&= \{ r \in R \text{ then } r + I = 0 + I \text{ and } r + J = 0 + J \} \\
&= \{ r \in R \text{ then } r \in I \text{ and } r \in J \} \\
&= I \cap J
\end{aligned}$$

$\mathbb{Z}_m \times \mathbb{Z}_n \cong \mathbb{Z}_{mn} \implies \mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic.

Let (a, b) be a generator of $\mathbb{Z}_m \times \mathbb{Z}_n$. Then $\text{order}((a, b)) = mn$.

Let $o_a = \text{order}(a)$ and $o_b = \text{order}(b)$.

$$\begin{aligned}
o_a &| m \wedge o_b | n \\
&\implies o_a, o_b | \text{lcm}(m, n) \\
&\implies \text{lcm}(m, n) \text{ is a common multiple of } o_a \text{ and } o_b \\
&\implies \text{lcm}(o_a, o_b) \leq \text{lcm}(m, n)
\end{aligned}$$

$$\text{order}((a, b)) = \text{lcm}(o_a, o_b) \leq \text{lcm}(m, n) = \frac{mn}{\text{gcd}(m, n)}$$

Therefore, $\text{gcd}(m, n) = 1$.

15. Let $\phi : \mathbb{Z} \rightarrow R$ be given by $\phi(n) = n1_R$. Show that ϕ is a ring homomorphism and $\text{Im } \phi$ is isomorphic to \mathbb{Z} or \mathbb{Z}_k for $k \in \mathbb{N}$.

Solution :- Here $\phi : \mathbb{Z} \rightarrow R$ be given by $\phi(n) = n1_R$

To show ϕ is a ring homomorphism.

Let, $\phi(m + n)$

$$\begin{aligned}
&= (m + n)1_R \\
&= m1_R + n1_R \\
&= \phi(m) + \phi(n)
\end{aligned}$$

Again, $\phi(mn)$

$$\begin{aligned}
&= mn1_R \\
&= (m1_R)(n1_R) \\
&= \phi(m)\phi(n)
\end{aligned}$$

Therefore, ϕ is a ring homomorphism.

$$\text{Im } \phi = \{ \phi(n) : n \in \mathbb{Z} \} = \{ n1_R : n \in \mathbb{Z} \}$$

Now we can define a function $f : \text{Im } \phi \rightarrow \mathbb{Z}$ as

$$f(n1_R) = n$$

This function is one one, onto and ring homomorphism.

Hence, $\text{Im } \phi \cong \mathbb{Z}$

16. Consider the maps $\phi_1 : \mathbb{R}[x] \rightarrow M_2(\mathbb{R})$ defined by $\phi(a_0 + a_1x + \dots + a_nx^n) = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}$ and

$\phi_2 : \mathbb{R}[x] \rightarrow M_3(\mathbb{R})$ defined by $\phi(a_0 + a_1x + \dots + a_nx^n) = \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{pmatrix}$. Show that ϕ_1, ϕ_2

are ring homomorphisms and find their kernels.

Solution :- Consider ,

$$\phi(a_0 + a_1x + \dots + a_nx^n) = \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix}$$

Then ,

$$\begin{aligned} & \phi\left((a_0 + a_1x + \dots + a_nx^n) + (b_0 + b_1x + \dots + b_nx^n)\right) \\ &= \phi\left((a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots + (a_n + b_n)x^n\right) \end{aligned}$$

$$= \begin{pmatrix} a_0 + b_0 & a_1 + b_1 \\ 0 & a_0 + b_0 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} + \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix}$$

$$= \phi\left((a_0 + a_1x + \dots + a_nx^n)\right) + \phi\left((b_0 + b_1x + \dots + b_nx^n)\right)$$

Again

$$\phi\left((a_0 + a_1x + \dots + a_nx^n) \times (b_0 + b_1x + \dots + b_nx^n)\right)$$

$$= \phi\left(a_0b_0 + (a_1b_0 + a_0b_1)x + \dots + a_nb_nx^{2n}\right)$$

$$= \begin{pmatrix} a_0b_0 & a_1b_0 + a_0b_1 \\ 0 & a_0b_0 \end{pmatrix}$$

$$= \begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} \times \begin{pmatrix} b_0 & b_1 \\ 0 & b_0 \end{pmatrix}$$

$$= \phi\left(a_0 + a_1x + \dots + a_nx^n\right) \times \phi\left(b_0 + b_1x + \dots + b_nx^n\right)$$

$\text{Ker } \phi = \{ a_0 + a_1x + \dots + a_nx^n \in \mathbb{R}[x] \text{ such that}$

$$\phi(a_0 + a_1x + \dots + a_nx^n) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \}$$

$\text{Ker } \phi = \{ a_0 + a_1x + \dots + a_nx^n \in \mathbb{R}[x] \text{ such that}$

$$\begin{pmatrix} a_0 & a_1 \\ 0 & a_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \}$$

$\text{Ker } \phi = \{ a_0 + a_1x + \dots + a_nx^n \in \mathbb{R}[x] \text{ such that } a_0 = a_1 = 0 \}$

$$\text{Ker } \phi = \{ 0 + 0 + a_2x^2 + \dots + a_nx^n \}$$

$$\text{Ker } \phi = \{ a_2x^2 + \dots + a_nx^n \text{ where } a_2, a_3, \dots, a_n \in \mathbb{R} \}$$

Similarly we can prove that

$$\phi(a_0 + a_1x + \dots + a_nx^n)$$

$$= \begin{pmatrix} a_0 & a_1 & a_2 \\ 0 & a_0 & a_1 \\ 0 & 0 & a_0 \end{pmatrix} \text{ is also a ring homomorphism.}$$

17. Show that following pairs of rings are not isomorphic.

- (a) the rings $\mathbb{Z}[\sqrt{2}]$ and $\mathbb{Z}[\sqrt{5}]$
- (b) the rings $\mathbb{Z}[\sqrt{-2}]$ and $\mathbb{Z}[\sqrt{-5}]$
- (c) $\mathbb{Z}_4 \times \mathbb{Z}_6$ and \mathbb{Z}_{24} .
- (d) \mathbb{R} and \mathbb{C} .

Solution :- (a)

$\mathbb{Z}(\sqrt{2})$ and $\mathbb{Z}(\sqrt{5})$ are not isomorphic.

$$\mathbb{Z}(\sqrt{2}) = \{ a + \sqrt{2}b, a, b \in \mathbb{Z} \} \text{ and}$$

$$\mathbb{Z}(\sqrt{5}) = \{ a + \sqrt{5}b, a, b \in \mathbb{Z} \}$$

Suppose $\mathbb{Z}(\sqrt{2})$ and $\mathbb{Z}(\sqrt{5})$ are isomorphic then

$$f(1) = 1$$

Then for $x \in \mathbb{Z}, x = (1 + 1 + 1 + \dots + 1)$ x times

$$f(x) = f(1) + f(1) + f(1) + \dots + f(1) \text{ x times}$$

$$= 1 + 1 + 1 + \dots + 1$$

$$= x \quad \forall x \in \mathbb{Z}$$

Since, f is a bijection, $f(\sqrt{2}) = \sqrt{5}$

$$\text{Now, } 2 = f(2) = f(\sqrt{2} \cdot \sqrt{2}) = f(\sqrt{2}) \cdot f(\sqrt{2})$$

$$2 = \sqrt{5} \times \sqrt{5} = 5$$

Which is not possible

Therefore, $\mathbb{Z}(\sqrt{2})$ and $\mathbb{Z}(\sqrt{5})$ are not isomorphic.

(b) Similarly we can prove that $\mathbb{Z}(\sqrt{-2})$ and $\mathbb{Z}(\sqrt{-5})$ are not isomorphic.

(c) $\mathbb{Z}_4 \times \mathbb{Z}_6$ is not isomorphic to \mathbb{Z}_{24} as $(4, 6) \neq 1$

(d)

Example. The ring \mathbb{R} is not isomorphic to \mathbb{C} . Indeed, assume $f : \mathbb{C} \rightarrow \mathbb{R}$ is an isomorphism. Remind that the element i in \mathbb{C} satisfies $i^2 = -1$. Applying f to both parts of the equality and taking into consideration the fact that $f(-1) = -f(1) = -1$ we get

$$-1 = f(-1) = f(i^2) = f(i) \cdot f(i).$$

Thus $f(i)$ is a real number whose square is -1 . But we know that there are no real numbers with this property.

18. Let R be a commutative ring and I be its ideal. Show that (i) $J_1 = \{x \in R : xa = 0 \forall a \in I\}$
(ii) $J_2 = \{x \in R : x^n \in I, \text{ for some } n \in \mathbb{N}\}$ are ideals of R .

Solution :- (i) Consider , $J_1 = \{x \in R : xa = 0 \forall a \in I\}$

Let , $x, y \in J_1$ then $xa = 0 = ya$ for all $a \in I$

As , $xa = 0 = ya$

Then $(x - y)a = 0$

$\Rightarrow (x - y) \in J_1$

Again , as R is a commutative ring then take $x \in J_1$ and $r \in R$ then $(rx)a = r(xa) = r \times 0 = 0$ as , $xa = 0$

So , $rx \in J_1$.

Similarly we can prove $xr \in J_1$. as R is a commutative ring

So , J_1 is an ideal of R .

(ii) Consider , $J_2 = \{x \in R : x^n \in I \text{ for some } n \in \mathbb{N}\}$

Let , $x, y \in J_2$ then $x^n, y^n \in I$

Since I is an ideal of R then $x^n, y^n \in I \Rightarrow x^n - y^n \in I$

And $(x - y)^n \in I$ as I is an ideal of R

Then , $x - y \in J_2$

Again for , $r \in R$ and $x \in J_2$ we get

$(rx)^n = r^n x^n \in I$ as I is an ideal of a commutative ring R .

So , rx and $xr \in J_2$

Therefore , J_2 is an ideal of R .

19. Find all the ideals of $\mathbb{Z}/12\mathbb{Z}$ using the correspondence theorem.

Solution :- Theorem 16 :-Correspondence theorem statement :-

16. Let $f : R \rightarrow S$ be an onto ring homomorphism and $K = \ker f$. Prove that there is one-one onto correspondence between ideals of R containing K and ideals of S .

The ideals of $\mathbb{Z}/12\mathbb{Z}$ are of the form $n\mathbb{Z}/12\mathbb{Z}$, where $n \in \mathbb{N}$ is such that $n \mid 12$.

Divisors of 12 are 1 , 2 , 3 , 4 , 6

Possible ideals :- $\mathbb{Z} / 12 \mathbb{Z} , 2\mathbb{Z} / 12\mathbb{Z} , 3\mathbb{Z} / 12 \mathbb{Z} , 4\mathbb{Z} / 12\mathbb{Z} , 6\mathbb{Z} / 12\mathbb{Z}$

20. Show that $\mathbb{Z}[i]/(2+i)$ is finite field, where $(2+i) = \{(2+i)(m+in) : m+in \in \mathbb{Z}[i]\}$.

Solution :-

$\phi : \mathbb{Z}[i] \rightarrow \mathbb{Z}_5$ is defined as
 $\phi(a + ib) = a - 2b \pmod{5}$

(i) To show that ϕ is well define

Suppose, $a + ib = c + id$

$$\Rightarrow a = c \text{ and } b = d$$

$$\Rightarrow a - 2b = c - 2d$$

$$\Rightarrow \phi(a + ib) = \phi(c + id)$$

ϕ is well define .

(ii) To show that ϕ is onto .

$$\phi(0 + i0) = 0 - 2 \times 0 = 0 \pmod{5}$$

$$\phi(1 + i0) = 1 - 2 \times 0 = 1 \pmod{5}$$

$$\phi(2 + i0) = 2 - 2 \times 0 = 2 \pmod{5}$$

$$\phi(3 + i0) = 3 - 2 \times 0 = 3 \pmod{5}$$

$$\phi(4 + i0) = 4 - 2 \times 0 = 4 \pmod{5}$$

Therefore, every element of \mathbb{Z}_5 has a pre-image in $\mathbb{Z}[i]$.

Therefore, ϕ is onto .

(ii) To show that ϕ is ring homomorphism.

$$\phi((a + ib) + (c + id))$$

$$= \phi((a + c) + i(b + d))$$

$$= \overline{(a + c) - 2(b + d)}$$

$$= \overline{(a - 2b) + (c - 2d)}$$

$$= \phi((a + ib)) + \phi((c + id))$$

$$\phi((a + ib) \times (c + id)) = \phi[(ac - bd) + i(ad + bc)]$$

$$= \overline{(a - 2d)(c - 2d)}$$

$$= \overline{(a - 2b)} \times \overline{(c - 2d)}$$

$$= \phi((a + ib)) \times \phi((c + id))$$

Hence, ϕ is a ring homomorphism.

$$\text{Ker } \phi = \{a + ib \mid \phi(a + ib) = \bar{0}\}$$

$$= \{a + ib \mid \overline{(a - 2b)} = \bar{0} \pmod{5}\}$$

$$= \{a + ib \mid a - 2b = 5k, k \in \mathbb{Z}\}$$

$$= \{(5k + 2b) + ib \mid b \in \mathbb{R}, k \in \mathbb{Z}\}$$

$$= \{b(2 + i) + (2 + i)(2 - i)k \mid b \in \mathbb{R}, k \in \mathbb{Z}\}$$

$$= \{(2 + i)(b + (2 - i)k) \mid b \in \mathbb{R}, k \in \mathbb{Z}\}$$

$$= \langle 2 + i \rangle$$

$\phi : \mathbb{Z}[i] \rightarrow \mathbb{Z}_5$ is a ring onto homomorphism with

$$\text{ker } \phi = \langle 2 + i \rangle$$

Therefore by first isomorphism theorem

$$\mathbb{Z}[i] / \langle 2 + i \rangle \cong \mathbb{Z}_5..$$

As \mathbb{Z}_5 is a field $\mathbb{Z}[i] / \langle 2+i \rangle$ is also a finite field as \mathbb{Z}_5 has finite number of elements.

21. Show that following rings of order 4 are non-isomorphic-

(a) \mathbb{Z}_4 (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\}$ (d) $\mathbb{Z}_2[x]/(x^2+x+1)$

Solution :- \mathbb{Z}_4 is not isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ as $(4, 2) \neq 1$

The set $G = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Z}_2 \right\} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\}$

But G is a Klein 4 group so G is non isomorphic to \mathbb{Z}_4 as \mathbb{Z}_4 is a cyclic group.

22. Show that $|\mathbb{Z}[i]/(z)| = n^2 + m^2$ where $z = n + mi \in \mathbb{Z}[i]$.

Solution :- Here, $\mathbb{Z}[i]/(z) = \{ (a+ib)+(z) , a + ib \in \mathbb{Z}[i] \}$

In (z) we get $\overline{n + mi} = \bar{0} \Rightarrow n = -mi$ in $\mathbb{Z}[i]/(z)$

$\Rightarrow n^2 = -m^2$

$\Rightarrow \overline{n^2 + m^2} = \bar{0} = \langle n + mi \rangle = (z)$ then $n^2 + m^2 \in (z)$

$\mathbb{Z}[i]/(z) = \{ (a+ib)+(z) , a + ib \in \mathbb{Z}[i] \}$

$= \{ c+(z) \mid c \in \mathbb{Z} \}$ since $n^2 + m^2 \in (z)$

The distinct elements of $\mathbb{Z}[i]/(z) = \{ c + (z) \mid 0 \leq c \leq n^2 + m^2 \}$

So number of elements of $|\mathbb{Z}[i]/(z)| = n^2 + m^2$

23. Show that aR is an ideal of R if and only if $ar = ra$ for all $r \in R$.

Solution :- Let $aR = \{ ar : r \in R \} = (a)$

Let aR be an ideal of R then

To show that $ar = ra$ for all $r \in R$

Let, $ar_1 \in aR$

But aR is an ideal of R then for any $r \in R$ we get

$r.ar_1, ar_1.r \in aR$

As, $r.ar_1 \in aR = (a)$ there exists $r_2 \in R$ such that

$r.ar_1 = ar_2$

So, $(ra) = ar_2r_1^{-1} = ar$

Conversely if $ar = ra$ then for any $x \in R$ we get $xar, arx \in aR$

Hence, aR is an ideal of R .

24. Let $\phi : \mathbb{R} \rightarrow M_{2 \times 2}(\mathbb{R})$ be given by $\phi(r) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$. Show that ϕ is a ring homomorphism and $\phi(1)$ is the multiplicative identity of $\text{Im } \phi$ but not of $M_{2 \times 2}(\mathbb{R})$.

Solution :- Consider, $\phi(r) = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}$

Then, $\phi(r+s) = \begin{pmatrix} r+s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} = \phi(r) + \phi(s)$

$$\text{Again } \phi(r \times s) = \begin{pmatrix} r \times s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} = \phi(r) \times \phi(s)$$

Therefore, ϕ is a ring homomorphism.

$$\phi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\mathbf{Im} \phi = \{ \phi(r), \text{ where } r \in \mathbb{R} \} = \left\{ \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix}, r \in \mathbb{R} \right\}$$

Hence, $\phi(1)$ is the multiplicative identity of $\mathbf{Im} \phi$ whereas the multiplicative identity of $M_{2 \times 2}(\mathbb{R})$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

25. Give example of a simple ring which is not a field.

Let's recall that a **simple ring** is a non-zero **ring** that has no two-sided **ideal** besides the zero ideal and itself. A **division ring** is a simple ring.

Is the converse true? *The answer is negative and we provide here a counterexample of a simple ring which is not a division ring.*

We prove that for $n \geq 1$ the matrix ring $M_n(F)$ of $n \times n$ matrices over a field F is simple. $M_n(F)$ is obviously not a division ring as the matrix with 1 at position $(1, 1)$ and 0 elsewhere is not invertible.