Prac 1 Objectives

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1. Let $H_1 = \{I, (12)\}$ and $H_2 = \{I, (123), (132)\}$. Then
   (a) H_1, H_2 are normal subgroups of S_3.
   (b) H_1 is a normal subgroup of S_3 but H_2 is not a normal subgroup of S_3.
   (c) H_1, H_2 are not normal subgroups of S_3
   ✓(d) H_2 is a normal subgroup of S_3 but H_1 is not a normal subgroup of S_3.

Soln:- $S_3 = \{I, (12), (23), (13), (123), (132)\}$

$H_1 = \{I, (12)\}$, $H_2 = \{I, (123), (132)\}$

$(123) \in S_3 \Rightarrow (123)^{-1} = (132)$

$(12) \in H_1$

$(123)(12)(123)^{-1}$

$= (123)(12)(132)$

$= (1)(23) \notin H_1 \Rightarrow H_1$ is not normal subgroup S_3

$(12) \in S_3 \Rightarrow (12)^{-1} = (12)$

$(123) \in H_2$

$\therefore (12)(123)(12)^{-1}$

$= (12)(123)(12)$

$= (132) \in H_2 \Rightarrow$ condition is satisfied

$(123) \in S_3 \Rightarrow (123)^{-1} = (132)$ and $(123) \in H_2$

$\therefore (123)(123)(123)^{-1}$

$= (123) I = (123) \in H_2 \Rightarrow$ condition is satisfied

$\therefore H_2$ is normal in S_3

2. Let $H_1 = \{\sigma \in S_n : \sigma(n) = n\}$, $H_2 = \{\sigma \in S_n : \sigma(k) = k,$ for some k, $1 \le k \le n\}$. Then
   (a) H_1, H_2 are normal subgroups of S_n.
   ✓(b) H_1 is a normal subgroup of S_n but H_2 is not a normal subgroup of S_n.
   (c) H_1, H_2 are not normal subgroups of S_n
   (d) H_2 is a normal subgroup of S_n but H_1 is not a normal subgroup of S_n.

Soln:- $H_1 = \{\sigma \in S_n : \sigma(n) = n\}$ $I = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 2 & 3 & 4 & \cdots & n \end{pmatrix}$

$= \{I\} \Rightarrow$ which is always subgroup & normal subgroup

$= \{ I \} \Rightarrow$ which is always subgroup & normal subgroup

$H_2 = \{ \sigma \in S_n : \sigma(k) = k$ for some k, $1 \leq k \leq n \}$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & k & \cdots & n \\ n & n-1 & n-2 & n-3 & & k & & 1 \end{pmatrix}$$

For, $n = 3$

$H_2 = \{ (12), (13), (23), I \}$, $(12) \in H_2$, $(123) \in S_3$

$(123)(12)(123)^{-1}$

$= (123)(12)(132)$

$= (1)(23)$

3. Let $G = \frac{\mathbb{Z}}{20\mathbb{Z}}, H = \frac{4\mathbb{Z}}{20\mathbb{Z}}$ (under addition). Then order of quotient group $\frac{G}{H}$ is

✓ (a) 4 (b) ∞ (c) 5 (d) 20

$$o(G) = o\left(\frac{Z}{20Z}\right) = \frac{o(Z)}{o(20Z)}$$

$$o(H) = o\left(\frac{4Z}{20Z}\right) = \frac{o(4Z)}{o(20Z)}$$

$$\therefore o\left(\frac{G}{H}\right) = \frac{o(G)}{o(H)} = \frac{\frac{o(Z)}{o(20Z)}}{\frac{o(4Z)}{o(20Z)}} = o\left(\frac{Z}{4Z}\right) = 4$$

$$\frac{Z}{4Z} = \{ 0+Z, 1+Z, 2+Z, 3+Z \}$$

4. Let H be a normal subgroup of G. Let $|aH| = 3$ in $\frac{G}{H}$ and $\circ(H) = 10$, then order of a is
(a) 1 ~~(b)~~ 30
(c) one of 3, 6, 15 or 30 (d) none of these.

$|aH| = 3$ in $\frac{G}{H}$, $O(H) = 10$

$$|aH| = \frac{O(a)}{O(H)} = 3 \qquad O\left(\frac{G}{H}\right) = \frac{O(G)}{O(H)}$$

5. Let G be a group of order 5. If $\Phi : \mathbb{Z}_{30} \to G$ is a group homomorphism, then ker Φ has order
✓ ~~(a) 5~~ (b) 30 or 6 (c) 30 or 5 (d) 1

order
(a) 5 (b) 30 or 6 (c) 30 or 5 (d) 1

$\phi : Z_{30} \to G$

$O(G) = 5$, $\ker \phi = \{ a \in Z_{30} ,\ \phi(a) = e \}$

e = identity of G

$O(Z_{30}) = 30$

As $\ker \phi$ is subgroup of $Z_{30} \Rightarrow$ order of $\ker \phi$ can be 1, 2, 3, 5, 6, 30

$\ker \phi$ is normal in Z_{30}

$\therefore$ Quotient group $= \frac{Z_{30}}{\ker \phi} \sim G$

6. Let G be a finite group. If $f_1 : G \to \mathbb{Z}_{10}$ and $f_2 : G \to \mathbb{Z}_{15}$ are onto group homomorphisms, then order of G is
(a) $30k$, where $k \in \mathbb{N}$ (b) 5^k, where $k \in \mathbb{N}$ (c) 10 or 15 (d) 5

As f_1 & f_2 both are onto

$\Rightarrow$ f_1 and f_2 both have pre-image

$\therefore$ $O(G)$ = multiple of lcm of order of Z_{10} & Z_{15}

$= [\text{lcm}\{10, 15\}]\, k$, k where $k \in N$

$= 30k$

7. In the quotient group $\frac{\mathbb{Z}_{18}}{<\bar{6}>}$ (under addition), the order of the element $\bar{5}+ <\bar{6}>$ is

(a) 5 (b) 6 (c) 2 (d) 3

$H = \langle \bar{6} \rangle = \{ 6, 12, 0 \}$

$5 \notin H$, $2(5 + H) = 10 + H$, $10 \notin H$

$3(5+H) = 15 + H$, $15 \notin H$

$4(5 + H) = 20 + H = 2 + H$, $2 \notin H$

$5(5 + H) = 25 + H = 7 + H$, $7 \notin H$

$6(5 + H) = 30 + H = 12 + H$, $12 \in H$

8. Let H be a subgroup of order 29 of a group G. If K is a subgroup of H, then

(a) K is abelian and normal subgroup of G.

(b) K is normal subgroup of H.

(c) K is cyclic but may not be a normal subgroup of H .

(d) H is normal subgroup of G and K is normal subgroup G.

Solution :-

Option (b)

H be a subgroup of order 29 of a group G . But 29 is a prime number . So if K be a subgroup of H then either O(K) = 1 or O(K) = 29 as by Lagrange's theorem we know O(K) / O(H)

Hence K is a trivial subgroup of H . All trivial subgroups of H are normal subgroup in H.

Therefore , K is normal in H.

9. Let $G = GL_2(\mathbb{R}), K = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}, H = \left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R}\right\}$. Then

(a) H is a normal subgroup of K and K is a normal subgroup of G.

(b) H is a normal subgroup of K but K is not a normal subgroup of G.

(c) H is a not normal subgroup of K but K is a normal subgroup of G.

(d) None of these.

Solution :- Option (b)

Here , G = $\mathbb{G}L_2(\mathbb{R})$

K = $\left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a , b , d \in \mathbb{R} , ad \neq 0 \right\}$ and H = $\left\{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \ b \in \mathbb{R} \right\}$

Let us calculate , $g\ h\ g^{-1}$ for h $\in H$ and g $\in G$

Take , h = $\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}$, $b_1 \in \mathbb{R}$ and $g = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} : a_2 , b_2 , d_2 \in \mathbb{R}$

Then , $g^{-1} = \frac{1}{a_2 d_2} \begin{pmatrix} d_2 & -b_2 \\ 0 & a_2 \end{pmatrix}$

Here , $g\ h\ g^{-1} = \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix} \frac{1}{a_2 d_2} \begin{pmatrix} d_2 & -b_2 \\ 0 & a_2 \end{pmatrix}$

$$= \frac{1}{a_2 d_2}\begin{pmatrix} a_2 & a_2 b_1 + b_2 \\ 0 & d_2 \end{pmatrix}\begin{pmatrix} d_2 & -b_2 \\ 0 & a_2 \end{pmatrix}$$

$$= \frac{1}{a_2 d_2}\begin{pmatrix} a_2 d_2 & -a_2 b_2 + a_2^2 b_1 + b_2 a_2 \\ 0 & a_2 d_2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{a_2 b_1}{d_2} \\ 0 & 1 \end{pmatrix} \in H \,, as \ \frac{a_2 b_1}{d_2} \in \mathbb{R}$$

Hence , $g\ h\ g^{-1} \in H.$

$\Rightarrow H$ is a normal subgroup of K.

$G = GL_2(\mathbb{R})$ = 2 × 2 invertible matrices forms group under multiplication.

To show that K is a normal subgroup of G.

$g^{-1}hg \in K$, for h $\in K$ and g $\in G$.

Let , h = $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ $, a, b\ , d \in \mathbb{R}\,, ad \neq 0$

Let , g ∈ $GL_2(\mathbb{R})$ such that g = $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in GL_2(\mathbb{R})$, where x , y , z , w ∈ ℝ .

$$ghg^{-1} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \frac{1}{xw - yz} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$= \begin{pmatrix} xa & xb + yd \\ za & zb + wd \end{pmatrix} \frac{1}{xw-yz} \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$$= \frac{1}{xw-yz} \begin{pmatrix} x^2a + xzb + zyd & xya + xwb + wyd \\ xza + z^2b + wzd & yzb + w^2d + zwb \end{pmatrix} \notin K$$

Therefore , K does not satisfy the condition of normal subgroup .

$\Rightarrow K$ is not a normal subgroup.

10. Let H be a normal subgroup of a finite group G. If $|H| = 2$ and G has an element of order 3 then

(a) G has a cyclic subgroup of order 6.

(b) G has a non-abelian subgroup of order 6.

(c) G has subgroup of order 4.

(d) None of these.

Solution :- Option (c)

H be a normal subgroup of a finite group G . Here | H | = 2. and G has an element of order 3.

Now let us consider the example :

G = S_3 = { I , (1 2) , (1 3) , (2 3) , (1 2 3) , (1 3 2) } O(G) = 6

Here , H = { I , (1 2) }

So we get H is normal in G and O(1 2 3) = 3 .

But S_3 is non abelian , non cyclic group.

11. Let G be a group of order 30. If $Z(G)$ has order 5, then

(a) $\frac{G}{Z(G)}$ is cyclic. (b) $\frac{G}{Z(G)}$ is abelian but not cyclic.

(c) $\frac{G}{Z(G)}$ is non-abelian. (d) None of these.

Solution :- Option (c)

G be a group of order 30 . If Z(G) has order 5 and we know

$$O\left(\frac{G}{Z(G)}\right) = \frac{O(G)}{O(Z(G))} = \frac{30}{5} = 6$$

$|G| = 30$, $|Z(G)| = 5$. $|G/Z(G)| = 6$

We know that all groups of order 6 are isomorphic to S_3 or $\mathbb{Z}/6\mathbb{Z}$.

well known result: If $G/Z(G)$ is cyclic, then G is abelian.

If $G/Z(G) \cong \mathbb{Z}/6\mathbb{Z}$,then G is abelian which is contradiction to $|Z(G)| = 5$. Therefore $G/Z(G) \cong S_3$

12. Let $G = GL_2(\mathbb{R})$, $H = \{A \in G : detA \in \mathbb{Q}\}$, then

(a) H is a normal subgroup of G. (b) H is not a subgroup of G.

(c) H is a subgroup which is not normal in G. (d) $H \subseteq Z(G)$.

Solution :- Option (a)

G = $GL_2(\mathbb{R})$ and H = $\{ A \in G : \det A \in \mathbb{Q} \}$

Here , Let M $\in G$ and A $\in H$ such that det A = a where a $\in \mathbb{Q}$

Consider , $M^{-1}AM$ thenn $\det(M^{-1}AM)$ = $\det(M^{-1})\det(A)\det(M)$

$= \det(M^{-1})\det(M).\det A$

= $\det(M^{-1}M)\det(A)$

= det I . det A

= det A

Therefore , $M^{-1}AM \in H \Rightarrow H$ is a normal subgroup of G.

13. Let $G = GL_2(\mathbb{R})$, $H = \{A \in G : detA = 2^m 3^n,$ for some $m, n \in \mathbb{Z}\}$, then

(a) H is a normal subgroup of G. (b) H is not a subgroup of G.

(c) H is a subgroup which is not normal in G. (d) $H \subseteq Z(G)$.

Solution :-

Option (a)

G = $GL_2(\mathbb{R})$ and H = $\{ A \in G : \det A = 2^m 3^n, for\ some\ m, n \in \mathbb{Z} \}$

Here , Let M $\in G$ and A $\in H$ such that det A = $2^m.3^n$,

where for some m , n $\in \mathbb{Z}$

Consider , $M^{-1}AM$ thenn $\det(M^{-1}AM)$ = $\det(M^{-1})\det(A)\det(M)$

$= \det(M^{-1})\det(M).\det A$

= $\det(M^{-1}M)\det(A)$

= det I . det A

= det A = $2^m . 3^n$, where for some m , n ∈ ℤ
Therefore , $M^{-1}AM \in H \Rightarrow \; H$ is a normal subgroup of G.

14. Let $G = U(16)$, $H = \{\bar{1}, \overline{15}\}$, $K = \{\bar{1}, \bar{9}\}$, then

(a) H, K are isomorphic groups and $\frac{G}{H}, \frac{G}{K}$ are isomorphic groups.

(b) H, K are not isomorphic groups but $\frac{G}{H}, \frac{G}{K}$ are isomorphic groups.

(c) H is not isomorphic to K.

(d) $\frac{G}{H}, \frac{G}{K}$ are not isomorphic groups.

Solution :-
Option (a)
G = U(16) , H = { $\bar{1}$, $\overline{15}$ } , $K = \{ \bar{1} , \bar{9} \}$,
Here H and K are cyclic group of order 2 . So they are isomorphic with each other .
Hence , $\frac{G}{H} \cong \frac{G}{K}$ *are* isomorphic groups .

15. Let $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in 2\mathbb{Z} \right\}, G = M_2(\mathbb{Z})$, under addition of 2×2 matrices. The quotient group $\frac{G}{H}$ has

(a) 4 elements (b) 16 elements (c) 12 elements (d) 8 elements

Solution :- Option (a)
Here , H = $\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b , c , \; d \in 2\mathbb{Z} \right\}, G = M_2(\mathbb{Z})$,
under addition 2 × 2 matrices
O(G) / O(H) = 4

16. Let $G = D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, a^4 = e = b^2, aba = b, H = \{e, b, a^2b, a^2\}, K = \{e, b\}$

(a) K is normal in H and H is normal in G. (b) K is not normal in H.

(c) K is normal in G. (d) H is not normal in G.

Solution :- Option (a)
G = $D_4 = \left\{ e , a , a^2, a^3, b , ab , a^2b , a^3b \right\}$ Here $a^4 = e = b^2$ and aba = b
Now , H = { e , b , a^2 , a^2b } , K = { e , b } Take , b ∈ H and b ∈ K
Then $b^{-1} = b \; as \; b^2 = e$, So , $b.b.b^{-1} = b.e = \; b \in K$
Take , $a^2b \in H$ and b ∈ K Then , $(a^2b)^{-1} = a^2b$
as , $(a^2b).(a^2.b)$ = a (aba)ab = (ab)(ab) = (aba)b = $b.b \; = b^2 = e$
$a^2b \, . \, b \, . \left(a^2b\right)^{-1} = a^2b.b.a^2b = a^2.b^2.a^2.b = a^2.e.a^2.b = a^4.b$
$= e.b = b \in K$
Take , $a^2 \in H$ and $b \in K$

Now as, $a^4 = e \Rightarrow a^2.a^2 = e \Rightarrow (a^2)^{-1} = a^2$

$a^2.b.(a^2)^{-1} = a^2ba^2 = a(aba)a = aba = b \in K$ as $aba = b$

Therefore we get K is a normal subgroup of G.

H = { e , b , a^2b , a^2}

And G = $\{e , a , a^2, a^3, b , ab , a^2b , a^3b \}$

Here , $a \times a^3 = e \Rightarrow (a^3)^{-1} = a$ and $a^{-1} = a^3$

$a^4 = e \Rightarrow a^2.a^2 = e \Rightarrow (a^2)^{-1} = a^2$

$b.b = e$ then $b^{-1} = b$

$ab\ .ab = (aba).b = b.b = b^2 = e \Rightarrow (ab)^{-1} = ab$

$(a^2b).(a^2.b)$ = a (aba)ab = (ab)(ab) = (aba)b = $b.b = b^2 = e$

Then , $(a^2b)^{-1} = a^2b$

Similarly , $(a^3b)^{-1} = a^3b$

(i) Take , b $\in H$, $a \in G$, $(a)^{-1} = a^3$

So , $aba^{-1} = aba^3 = (aba)a^2 = ba^2 = a^2b \in H$,

(ii) take b $\in H$ and $a^3b \in G$ and $(a^3b)^{-1} = a^3b$

So , (a^3b) $b\ (a^3b)^{-1} = a^3b.b.a^3b = a^3.b^2.a^3b$

$= a^6.b = a^4.a^2.b = e.a^2b = a^2b \in H,$

Take , $a^2b \in H$ and $ab \in G$

$(ab)^{-1} = ab$

Then , $ab(a^2b)(ab)^{-1} = ab(a^2b)ab = (aba)(ab)ab$

$= b(aba)b = b.b.b = b \in H$,

Similarly we can show that for all other elements of G and H .

Therefore H is normal in G .

To show that K is not normal in G .

K = { e , b } , G = $\{e , a , a^2, a^3, b , ab , a^2b , a^3b \}$

Take b $\in K$ and $a^3b \in G$ where $(a^3b)^{-1} = a^3b$

Hence , $(a^3b)b\ (a^3b)^{-1} = a^3b.b.a^3b = a^3.b^2.a^3b$

$= a^3.e.a^3.b = a^4.a^2.b = e.a^2.b = a^2b \notin K$

Therefore K is not normal in G .

17. The quotient group $\left(\frac{\mathbb{Q}}{\mathbb{Z}}, +\right)$ is

(a) an infinite group in which only identity is of finite order.
(b) is an infinite cyclic group of finite index.

(c) an infinite group in which every element is of finite order.
(d) None of these.

Solution :- Option (c)

Example of an Infinite Group Whose Elements Have Finite Orders

Choose some $\alpha \in \mathbb{Q}/\mathbb{Z}$. By definition of the quotient group, there is a rational number r for which

$$\alpha = r + \mathbb{Z}$$

Now by the definition of $\mathbb{Q}$, there are integers a and $b \neq 0$ (we take $b > 0$) without any loss of generality) for which

$$r = \frac{a}{b} \implies \alpha = \frac{a}{b} + \mathbb{Z}$$

Therefore,

$$b\alpha = b\left(\frac{a}{b} + \mathbb{Z}\right) = b\frac{a}{b} + \mathbb{Z} = a + \mathbb{Z}$$

But since a is an integer, $a + \mathbb{Z} = \mathbb{Z}$ is the identity in $\mathbb{Q}/\mathbb{Z}$, so α has finite order (in fact, the order is at most b) as desired.

18. Let G be a non-Abelian group of order pq, where p and q are distinct primes then
(a) $o(Z(G)) = p$ (b) $o(Z(G)) = q$
(c) $Z(G) = \{e\}$ (d) None of these.

Solution :- Option (c)
O(G) = pq , which is non abelian where p and q are distinct prime.
Then , Z(G) = { e }

19. If H is any non-trivial sugroup of a cyclic G then G/H

(a) is infinite if G is infinite.
(b) is finite
(c) is not cyclic
(d) None of these.

Solution :- Option (a)
As H is any non-trivial subgroup of a cyclic G then G / H is infinite if G is infinite .

20. If G be an Abelian group then $H = \{(g,g) : g \in G\}$ is

(a) normal in $G \times G$.

(b) is not normal in $G \times G$.

(c) is not a subgroup of $G \times G$

(d) None of these.

Solution :- Option (a)

G be an abelian group then H= { (g , g) : g ∈ G } is a normal subgroup in G × G .

21. The index of centre of a finite non-Abelian group

(a) is $o(G)$.

(b) is a prime

(c) can not be a prime

(d) None of these.

Solution :-

Option (b)

By the statement of the theorem we know :

The index of Centre of a finite non-abelian group is a prime number.

22. If N is a normal subgroup of G and all the elements of G/N and N have finite order, then

(a) every element of G has finite order.

(b) every element of G has infinite order.

(c) G can have elements of infinite order.

(d) None of these.

Solution :- Option (a)

If every element of G/H has finite order and every element of H has finite order, then every element of G has finite order

Let $g \in G$. Consider the coset gH. Since every element in G/H has finite order, there exists $n \in \mathbb{N}$ such that $g^n H = (gH)^n = eH = H$, i.e. $g^n \in H$. Now we use the fact that every element in H has finite order, so there exists $m \in \mathbb{N}$ such that $(g^n)^m = e$, i.e. $g^{mn} = e$. Thus g has finite order.

23. If H is a subgroup of S_n having order $n!/2$, then which of the following is not true

(a) H is normal in S_n

(b) $\sigma^2 \in H$ for every $\sigma \in S_n$.

(c) H contains all 3-cycles.

(d) $H \neq A_n$.

Solution :- Option (a)

We know the subgroup with order n!/ 2 is the Alternating group A_n of $S_n. Hence, H = A_n$ *And we know*

The alternating group is a normal subgroup of the symmetric group