Practical −2.1

Example and Properties of Groups

Objective Questions

1) Which of the following set is a group under indicated binary operation,

(a) $(\mathbb{N}, +)$ (b) $(\mathbb{R}, \cdot)$

(c) $(\mathbb{R}^*, +)$ (d) $(\mathbb{R}, +)$

2) Let G be a group and $a_1, a_2, a_3, a_4 \in G$ then inverse of $a_1 {a_2}^{-1} {a_3}^{-1} a_4$ is

(a) ${a_1}^{-1} a_2 a_3 {a_4}^{-1}$ (b) ${a_4}^{-1} a_2 a_3 {a_1}^{-1}$

(c) ${a_1}^{-1} a_3 a_2 {a_1}^{-1}$ (d) None of these

3) Let G be a group and $a, b, c \in G$ then the solution of the equations $axb^{-1} = c$ and $a^{-1}y^{-1}b^{-1} = c$ are

(a) $x = a^{-1}c^{-1}b, y = a^{-1}c^{-1}b^{-1}$ (b) $x = a^{-1}cb, y = b^{-1}c^{-1}a^{-1}$

(c) $x = a^{-1}c^{-1}b, y = b^{-1}c^{-1}a^{-1}$ (d) $x = a^{-1}c^{-1}b, y = b^{-1}c^{-1}a^{-1}$

4) Let O denotes the set of odd integers. Then

(a) O forms group under multiplication (b) O forms group under addition

(c) O dosen't forms group under addition (d) None of these

5) $(\mathbb{N}, +)$ is not a group as

(a) Associative law does not hold in $(\mathbb{N}, +)$ (b) Cancellation law does not hold in $(\mathbb{N}, +)$

(c) $\mathbb{N}$ has no identity element with respect to + (d) $\mathbb{N}$ has elements which have more than one inverse with respect to $+$

6) Consider the set $G = \{\bar{5}, \overline{15}, \overline{25}, \overline{35}\}$ under multiplication of residue classes modulo 40. Then

(a) G is not a group as $\bar{1} \notin G$ (b) G is a group with $\overline{25}$ as identity

(c) G is a group with $\bar{5}$ as identity (d) None of these

7) Consider the group $(\mathbb{Z}, *)$, where $\mathbb{Z}$ is set of integers and $a * b = a + b - 6$

(a) -6 is identity of $(\mathbb{Z}, *)$ and inverse of $a \in \mathbb{Z}$ is $6 - a$.

(b) 6 is identity of $(\mathbb{Z}, *)$ and inverse of $a \in \mathbb{Z}$ is $12 - a$.

(c) 6 is identity of $(\mathbb{Z}, *)$ and inverse of $a \in \mathbb{Z}$ is $6 - a$.

(d) None of these

8) Consider the sets (i) $(\mathbb{Z}, \cdot)$(ii) $(\mathbb{N}, +)$(iii) $(\mathbb{R}, \cdot)$
(iv) $(G, \cdot)$ where $G = \{2^m 3^n : m, n \in \mathbb{Z}\}$

(a) (i) and (iv) are groups. (b) Only (iv) is group.

(c) (ii) and (iii) are groups. (d) None of these

9) Which of the following group is non−Abelian

(a) $(\mathbb{Q}, +)$ (b) S_3

(c) μ_4, the fourth root of unity (d) None of these

10) The identity of $G = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{Q}, a \neq 0\right\}$ under multiplication of 2×2 matrices is

(a) $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ (c) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

11) Let $G_1 = \{\bar{1}, \bar{2}, \bar{3}\}$ under multiplication of residue classes modulo 4 and $G_2 = \{\bar{1}, \bar{2}\}$ under multiplication of residue classes modulo 3. Then

(a) G_1 and G_2 are groups (b) G_1 is a group but G_2 is not a group

(c) G_1 is not a group but G_2 is a groups (d) None of these

12) Let $G = \{\bar{1}, \bar{3}, \bar{9}, \overline{11}, \bar{a}, \overline{19}, \overline{27}, \overline{33}\}$ under multiplication of residue classes modulo 40 where $1 \leq a \leq 40$. Then

(a) $a = 13$ (b) $a = 17$ (c) $a = 23$ (d) $a = 31$

13) Let X be a non-empty set and $P(X)$ denoted power set of X. For the group $(P(X), \Delta)$ such that $A \,\Delta\, B = A - B$

(a) X is identity and for $A \in P(X), A^{-1} = A$ (b) $\emptyset$ is identity and for $A \in P(X), A^{-1} = A$

(c) $\emptyset$ is identity and for $A \in P(X), A^{-1} = A^c$ (d) None of these

14) Let $G = GL_n(\mathbb{R})$. Then

(a) G is an infinite Abelian group for $n \geq 2$ (b) G is a finite Abelian group for $n = 2$

(c) G is an infinite non-Abelian group for $n \geq 2$ (d) None of these

15) Let G be a group $a, b \in G$ such that $ab = ba, o(a) = m, o(b) = m$. Then

(a) $o(ab) = mn$ (b) $o(ab) = l.c.m[m, n]$

(c) $o(ab)$ divides l.c.m$[m, n]$ but may not be equal to $l.c.m[m, n]$ (d) None of these

16) Let G be a group, $a \in G$ a such that $o(a) = n$. If $a^r = a^s$ then

(a) $r = s$ (b) $r + s = n$ (c) $r \equiv s \bmod n$ (d) None of these

17) The number of elements in $U(n)$, the group of prime residue class modulo n is

(a) n (b) $n - 1$ (c) $\phi(n)$ (d) None of these

18) The set $\mathbb{Z}^*{}_n$ under multiplication of residue class modulo n is a group if and only if

(a) n is odd (b) n is prime (c) n is even (d) None of these

19) In the group $(\mathbb{Z}_{42}, +)$, order of $\overline{18}$ is

(a) 7 (b) 18 (c) 6 (d) None of these

20) The inverse of $\bar{3}$ in the group $(\mathbb{Z}^*{}_7, \cdot)$ is

(a) $\bar{3}$ (b) $\bar{5}$ (c) $\bar{2}$ (d) None of these

21) Which of the following is a Klein-4 group.

(a) $U(10)$ (b) μ_4, fourth root of unity (c) $U(8)$ (d) None of these

22) Let G be a group and $a, b, \in G$. Let e be the identity element of G. If $a^4 = e, ab = ba^2$ then

(a) $a = e$ (b) $a = b$ (c) $a = b^2$ (d) $b = a^2$

23) Let G be a finite group. Then the number of solutions of the equation $x^3 = e$

(a) is one (b) is a multiple of 3

(c) is always odd number (d) is always even number

24) Consider the following statements.
(i) In a group G, the equation $x^2 = x$ has two solutions.
(ii) In a group G, the equation $x^2 = e$ may have more than two solutions.
(iii) In a group G, if $x^3 = e$ has more than one solution then the number of solutions is an even number.
(iv) In a group G, the equation $ax = b$, where $a, b \in G$ can have more than one solution.
Then

(a) (ii) and (iii) are true statements (b) only (ii) is true

(c) (ii) and (iv) are false (d) None of these

25) Let G be a group and $a \in G$. If $o(a) = 20$, then $o(a^4)$ is

(a) 15 (b) 5 (c) 12 (d) 20

26) Consider the group (Q^+, o) where Q^+ is set of positive rational numbers and $a\ o\ b = \frac{ab}{3}$ for $a, b \in Q^+$. Then

(a) $\frac{1}{3}$ is the identity element of the group and for $a \in Q^+, a^{-1} = \frac{1}{3a}$

(b) 1 is the identity element of the group and for $a \in Q^+, a^{-1} = \frac{3}{a}$

(c) 3 is identity element of the group and $a^{-1} = \frac{3}{a}$ for $a \in Q^+$

(d) None of the above

27) Let G be a group and $a \in G$. If $o(a) = 17$, then $o(a^8)$ is

(a) 17 (b) 16 (c) 8 (d) 5

28) Suppose a group G contains elements a and b such that $o(a) = 4, o(b) = 2, a^3b = ba$. Then $o(ab)$ is

(a) 2 (b) 5 (c) Infinite (d) 6

29) In a group G, the number of elements $a \in G$ such that $a^2 = a$ is

(a) 0 (b) 1 (c) 2 (d) None of these

Descriptive Questions

1) Show that $(\mathbb{Z},*)$ is a group, where $a * b = a + b - 5$, for $a, b \in \mathbb{Z}$.

2) Show that $(\mathbb{Q}^*, o)$ is a group, where $aob = \frac{ab}{6}$, for $a, b \in \mathbb{Q}^*$.

3) Show that $(\mathbb{R} - \{1\},*)$ is a group, where $a * b = a + b - ab$for $a, b \in \mathbb{R} - \{1\}$. Hence solve $2 * x * 3 = 7$in $\mathbb{R} - \{1\}$.

4) Show that the following are group under multiplication of 2×2 matrices,

(i) $G = \left\{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{R}, a \neq 0\right\}$

(ii) $G = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a \in \mathbb{R}, a \neq 0\right\}$

(iii) $G = \left\{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, b, d \in \mathbb{R}, ad \neq 0\right\}$

(iv) $G = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{Q}, a \neq 0\right\}$

(v) $G = \left\{\begin{bmatrix} a0 \\ b1 \end{bmatrix} : a, b \in \mathbb{R}, a \neq 0\right\}$

(vi) $G = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix} : a, b \in \mathbb{R}, aandbnotbothzero\right\}$

5) Show that $G = \left\{\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} | a, b, c, \in \mathbb{R}\right\}$ is a group under multiplication of 3×3 real matrices.

6) Construct composition table for the following sets and show they forms group,

a) $(\mathbb{Z}_6, +)$under addition of residue classes modulo 6.

b) $({\mathbb{Z}^*}_7, \cdot)$under multiplication of residue classes modulo 7.

c) $U(10)$,under multiplication of residue classes modulo 10.

d) $U(12)$under multiplication of residue classes modulo 12.

e) $U(14)$under multiplication of residue classes modulo 14.

7) Construct composition table for the following sets and show they forms group.

Which of them are Klien-4 group Justify?

a) $G = \{1, -1, i, -i\}$ under multiplication of complex numbers

b) $U(8)$, under multiplication of residue classes modulo 8.

c) $G = \{\overline{5}, \overline{15}, \overline{25}, \overline{35}\}$under multiplication of residue classes modulo 40.

d) $G = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\right\}$under multiplication of 2×2 matrices.

8) Write all complex roots of $x^7 = 1$. Show that they form a group under usual multiplication.

9) Show that the set of all complexes numbers $a + ib$ such that $a^2 + b^2 = 1$ is a group under the usual multiplication of complex numbers.

10) Let G be a group of order 4. If e, a, b are distinct elements of G such that $a^2 = e, b^2 = e$, where e is the identity element of G, then show that $G = \{e, a, b, ab\}$ and form composition table of G.

11) If $(G, \cdot)$ is a group, then show that $(G^0, *)$ is a group where $G^0 = G$ and for $a, b \in G, a * b = b \cdot a$

12) Let U be a non-empty set. $G = \{X | X \subseteq U\}$. Show that G is an Abelian group under binary operation $\oplus$defined by $X \oplus Y = (X - Y) \cup (Y - X)$for $X, Y \in G$.

13) In a group G prove the following:

a) For $a, b, c \in G$. Show that there exists a unique element x in G such that $axb = c$.

b) $G = \{a_1, a_2, \dots, a_n\}$. Let $a_1 a_2 \dots a_n = x$. Prove that $xx = e$.

c) If $a, b \in G$ and $a^5 = b^5, a^{12} = b^{12}$. Then show that $a = b$.

d) Let $a, b, \in G$. Suppose that $a^2 = e$ and $ab^4a = b^7$. Prove that $b^{33} = e$.

e) Let G be a group such that $ab = ca \Rightarrow b = c$, for $a, b, c \in G$. Show that G is Abelian.

f) Let G be a group such that $(ab)^3 = a^3b^3, (ab)^4 = a^4b^4, (ab)^5 = a^5b^5 \forall a, b \in G$. Show that G is an Abelian group.

g) Let G be a group such that $a^5b^5 = b^5a^5, a^{12}a^{12} = b^{12}a^{12}, \forall a, b \in G$. Show that G is Abelian.

h) Let G be agroup and $a, b \in G$ such that $ab = ba$. Prove that $(ab)^n = a^nb^n \forall n \in \mathbb{N}$.

14) In $GL(2, \mathbb{R})$, show that $A = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$ and $b = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ are elements of finite order, whereas AB is of infinite order.

15) Find the order of $\overline{12}, \overline{16}, \overline{18}$ in each of the following group. Also state clearly the result used. a) $(\mathbb{Z}_{28}, +)$ (b) $(\mathbb{Z}_{40}, +)$ (c) $(\mathbb{Z}_{42}, +)$

16) Compute o($\overline{2}$)in $U(7)$, $U(11)$, $U(41)$.

17) Let G is a group of order 8, show that G has an element of order 2.

18) Consider the group $\mathbb{Z}_{15}$. Find the smallest positive integer n such that $n\overline{5} = \overline{0}$in $\mathbb{Z}_{15}$.

19) Consider the group $\mathbb{Z}_{20}$. Find the smallest positive integer n such that $n\overline{5} = \overline{0}$in $\mathbb{Z}_{20}$.

20) Find the smallest positive integer n such that $\overline{7}^n = \overline{1}$ in $U(10)$ and in $U(12)$.

21) Find the order of $\overline{6}$ in the group $\mathbb{Z}_{10}$ and the order of $\overline{3}$ in $U(10)$.

22) In the group $\mathbb{Z}_{15}$, Find the order of the following elements: $\overline{5}, \overline{8}$ and $\overline{10}$.

23) Let G be a group and $a \in G$. If $o(a) = 24$ then find $o(a^4)$, $o(a^7)$ and $o(a^{10})$.

24) In the group $\mathbb{Z}_{18}$, Find the order of the elements: $\overline{7}$, $\overline{10}$ and $\overline{12}$.

Practical No : 2.2
Group of Symmetry and symmetric group S_n

Objective Questions

(1) The group of symmetries of a rectangle which is not a square is

(a) $(\mathbb{Z}_4, +)$ (b) V_4 −Klein's Four group (c) D_4 (d) None of these

(2) The group of symmetries of a square has order

(a) 4 (b) 24 (c) 8 (d) None of these

(3) The group of symmetries of

(a) a square is Abelian (b) an equilateral triangle is Abelian

(c) a rectangle is Abelian (d) None of these

(4) The group of symmetries of a regular n −gon $(n > 3)$ has

(a) n elements of order 2 and $n - 1$ elements of order n.

(b) $n$elements of order 2 if n is odd.

(c) exactly 2 elements of order n.

(d) None of the above.

(5) Let G be the group of symmetries of a square. Then center of G has

(a) only one element. (b) four elements.

(c) exactly two elements. (d) None of these

(6) Let G be the group of symmetries of a regular pentagon. Then G has

(a) 5 reflections and 5 rotations. (b) no reflections and 10 rotations.

(c) 10 reflections and 10 rotations. (d) None of these

(7) Let $\sigma \in S_n$ be a cycle of length r. Then σ^k is a cycle of length r if

(a) k is odd (b) $(k, r) = 1$ (c) k is even (d) None of these

(8) The number of elements of order 2 in S_4 is

(a) 8 (b) 6 (c) 9 (d) None of these

(9) Let $H = \{\alpha \in S_5 : \alpha(1) = 1\}$. Then

(a) H has only identity element. (b) H has only even permutations.

(c) $o(H) = 24$ (d) $o(H) = 12$

(10) The orders of elements in the group A_4 are

(a) 1 , 2 and 3. (b) 1 , 2 and 4. (c) 1 , 3 and 4. (d) 1 , 2.

(11) The maximum order of an element in S_{10} is

(a) 10 (b) 24 (c) 25 (d) 30

(12) The normalizer $N((123))$ of the element (123) in S_3 is

(a) $\{I, (123)\}$ (b) $\{I, (123), (132)\}$

(c) $\{I, (132)\}$ (d) None of these

(13) For what value of n, $D_n = S_n$

(a) $n = 3$ (b) $n = 4$ (c) $n = 5$ (d) there is no such n exists

(14) If α and β are disjoint permutations in S_n such that $o(\alpha) = m$, $o(\beta) = k$ then order of $(\alpha \circ \beta)$ is

(a) always less than n. (b) mk (c) $l.c.m[m, k]$ (d) None of these

(15) If $\sigma \in S_n$ has odd order then

(a) σ is odd permutation. (b) σ is even permutation.

(c) σ may be even or odd permutation. (d) None of these

(16) Every permutation in A_n can be written as product of

(a) even number of transpositions (b) p transpositions, where p is an odd prime

(c) odd number of transpositions (d) None of the above

(17) Let $\beta \in S_7$ such that $\beta^4 = (2143567)$ then β is equal to

(a) (2516473) (b) (2457136) (c) (2631754) (d) None of these

(18) In the group of symmetry of a rectangle (which is not a square) , the element obtain in rotation of rectangle by angle $180°$ is

(a) $(12)(34)$ (b) $(13)(24)$ (c) $(14)(23)$ (d) None of these

(19) The group of symmetric of an equilateral triangle is

(a) S_3 (b) A_3 (c) A_4 (d) S_4

(20) In the group of symmetries of an equilateral triangle with vertices. 1, 2,3, let σ denote clockwise rotation through 120° about center and τ denote reflection about the line joining vertex 1 and mid point of opposite side then

(a) σ does not commute with τ. (b) σ commutes with every element of the group

(c) σ, τ have order 2 (d) σ, τ have order 3.

(21) Let G_1, G_2, G_3 denote the group of symmetries of a equilateral rectangle, rectangle and square respectively. Then,

(a) G_1, G_2, G_3 are all abelian (b) G_1, G_2, G_3 are all non-abelian

(c) G_1 and G_3 are non-abelian and G_2 is abelian (d) G_1, G_2 are abelian and G_3 is non-abelian

(22) Let G denote the group of symmetries of a square then the number of elements in the centre of G is

(a) 1 (b) 2 (c) 3 (d) 4

(23) The group of symmetries of a regular $n - gon$ is

(a) S_n (b) S_{2n} (c) $\mathbb{Z}_n$ (d) D_n

(24) Consider the groups A_3, A_4, S_3, S_4. Among them

(a) A_3 is abelian, S_3, S_4 are non-abelian (b) A_3, A_4 are abelian, S_3, S_4 are non-abelian

(c) All the groups are non-abelian (d) A_3, S_3 are abelian, A_4, S_4 are non-abelian

(25) S_3 has no element of order

(a) 1 (b) 2 (c) 4 (d) 3

Descriptive Questions

(1) List all elements of group of symmetries of an equilateral triangle as permutations on its vertices and form the composition table.

(2) List all elements of group of symmetries of a square as permutations on its vertices. Find elements σ , τ in G such that $\sigma^4 = e = \tau^2$, $\tau\sigma = \sigma^3\tau$ and $G = \langle\sigma , \tau\rangle$

(3) List all elements of group of symmetries of a rectangle (which is not a square) as permutations on its vertices and form the composition table. Show that this group is Klein's four group.

(4) List all elements of S_3 and S_4. Express them as a product of disjoint cycles and determine which are even and which are odd. Further list the elements of A_3 and A_4 .

(5) Find the order of each element of S_3 , S_4 , A_3 and A_4 .

(6) Let $\alpha = (124)(157)$, $\beta = (162)(2415)(135)$. Find α^{2018}, β^{2019}. Which of these are cycles ?

(7) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 4 & 2 & 1 \end{pmatrix}$, $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 4 & 1 & 5 & 3 \end{pmatrix} \in S_6$. Find $\alpha\beta$, $\alpha\beta^{-1}$, $\alpha^2\beta$. Express them as a product of disjoint cycles and find their orders in the group S_6 .

(8) Show that there is no $\alpha \in S_3$ such that $\alpha^{-1}(123)\alpha = (12)$.

(9) Find all positive integers n such that S_5 and S_6 contains an element of order n .

(10) i) In S_5 , Find α and β such that $o(\alpha) = 2$, $o(\beta) = 2$ but $o(\alpha\beta) = 5$.

ii) In S_5 , Find α and β such that $o(\alpha) = 3$, $o(\beta) = 3$ but $o(\alpha\beta) = 5$.

(11) i) Let $\sigma = (123 \ldots\ldots 10)$. Find σ^k for $0 \leq k \leq 9$ and find the integers for which σ^k is a $10-$cycle.

ii) Let $\beta = (12)(34)(56)(78)(9\ 10)$. Determine whether there exists a $10-$cycle α such that $\alpha^k = \beta$ for some $k \in \mathbb{N}$.

(12) i) Show that A_8 contains an element of order 15. Also find an element of order 15 in A_8 .

ii) Let $\alpha = (13579)(246)(8\ 10)$. If α^m is a $5-$cycle, what can be said about m.

(13) Suppose that n is odd and $\sigma \in S_n$ is a n $-$cycle. Prove that σ does not commute with any element of order 2.

(14) Find normalizer of each element of S_3 , where normalizer of any $a \in G$ is defined as

$N(a) = \{x \in G : ax = xa\}$.

(15) If α and β are disjoint permutations in S_n such that $o(\alpha) = m$, $o(\beta) = k$ then show that order of $(\alpha \circ \beta)$ is $l.c.m[m, k]$.

(16) Find $a, b \in S_3$ such that $(ab)^2 \neq a^2b^2$ and $(ab)^{-1} \neq a^{-1}b^{-1}$.

(17) Let $\alpha = (125)(6135)$ and $\beta = (134)(265)(243)$. Write α and β as a product of disjoint cycles.

i) Find α^{-1}, β^{-1}, $\alpha\beta$, $\beta\alpha$, $\alpha\beta\alpha^{-1}$.

ii) Verify that $o(\alpha^{-1}) = o(\alpha)$, $o(\alpha\beta) = o(\beta\alpha)$, $o(\alpha\beta\alpha^{-1}) = o(\beta)$.

iii) Verify that $(\alpha\beta\alpha^{-1})^{12} = \alpha\beta^{12}\alpha^{-1}$.

(18) Let $\alpha = (126)(634)(562)$ and $\beta = (154)(3124)$. Write α and β as a product of disjoint cycles.

i) Find α^{-1}, β^{-1}, $\alpha\beta$, $o(\alpha)$, $o(\beta)$.

ii) Is $o(\alpha\beta) = o(\alpha)o(\beta)$? Justify.

iii) Find $o(\alpha^{16})$, $o(\beta^{20})$.

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Practical no. -2.3

Subgroups, Order of an Element

Objective Questions

1. Let H be a proper subgroup of the group $(\mathbb{Z},+)$ such that $18, 30, 40$ belong to H then
(a) $H = 10\mathbb{Z}$ (b) $H = 2\mathbb{Z}$ (c)$H = \mathbb{Z}$ (d) $H = 360\mathbb{Z}$

2. Let G be a group having elements of order $1, 2, 3, 4, 5, 6$. The minimal possible order of G is
(a) 100 (b) 30 (c) 60 (d) 1

3. Let G be a group of odd order and e be an identity element of G. The equation $x^2 = e$ has
(a) a unique solution in G (b) no solution in G (c) two solutions in G (d) Cannot say

4. Let G be a non-Abelian group $Z(G) = \{x \in G \mid ax = xa \ \forall \ a \in G\}$ then
(a) $Z(G) = \{e\}$ (b) $Z(G) \neq G$ and $Z(G)$ is Abelian (c) $Z(G) = G$ (d) $Z(G)$ is non-Abelian

5. A group G has subgroups of order 45 and 75. If $0(G) < 400$ then
(a) $0(G) = 150$ (b) $0(G) = 225$ (c) $0(G) = 375$ (d) $0(G) = 150$ or 225

6. A_4 has no subgroup of order
(a) 2 (b) 3 (c) 4 (d) 6

7. Let $\alpha = (13\ )(24) \in S_4$. Let $N(\alpha) = \{\sigma \in S_4 \mid \sigma\alpha = \alpha\sigma\}$. Then order of subgroup $N(\alpha)$is
(a) 4 (b) 6 (c) 12 (d) 24

8. Let $\alpha = (1\ 2\ 3) \in S_3$ .Let $N(\alpha) = \{\sigma \in S_3 \mid \sigma\alpha = \alpha\sigma\}$. Then order of subgroup $N(\alpha)$ is
(a) 3 (b) 2 (c) 1 (d) 6

9. Let p, q be distinct prime integers. H be a proper subgroup of $\mathbb{Z}$ under addition. If H contains exactly 3 elements from the set $\{p, pq, p+q, p^q, q^p\}$ then the 3 elements of H are
(a) $pq,\ p^q,\ q^p$ (b) $p+q,\ pq,\ p^q$ (c) $p,\ p^q,\ q^p$ (d) $p,\ pq,\ p^q$

10. $\mathbb{C}^*$ under multiplication has
(a) No non-trivial finite cyclic subgroup
(b) Only one non-trivial finite cyclic subgroup
(c) Infinitely many non-trivial finite cyclic subgroups
(d) None of these.

11. Let p be prime. If a group G has more than $p-1$ elements of order p, then

(a) G is cyclic
(b) G is not cyclic
(c) G has a unique subgroup of order p.
(d) None of these.
(e)

12. $U(12)$, the multiplicative group of units in $\mathbb{Z}_{12}$, has

(a) Only one proper subgroup
(b) Two proper subgroup.
(c) Three proper subgroups
(d) None of these.

13. $\{\bar{1}, \bar{4}, \overline{7}\}$ in $U(9)$ is

(a) Not a subgroup of $U(9)$
(b) Is a subgroup of $U(9)$
(c) Is a cyclic subgroup of $U(9)$
(d) None of these.

14. $\{\sigma \in S_6 : \sigma(6) = 1\}$ in S_6 is
(a) Not a subgroup of S_6
(b) Is a subgroup of S_6
(c) Is a cyclic subgroup of S_6
(d) None of these.

15. If a group G has only one element a of order n , then which of the following is true;
(i) $a \in Z(G)$ (ii) $n = 2$ (iii) a is inverse of itself
(a) Only (i)
(b) Only (ii) and (iii)
(c) All the three
(d) None of these.

Descriptive Questions

1. In each case, determine whether H is a subgroup of the group G under usual operation.
   (a) $H = \{3n | n \in \mathbb{Z}\}, G = \mathbb{Z}$
   (b) $H = \{n | n \in \mathbb{Z} \text{ and } n \geq 0\}, G = \mathbb{Z}$
   (c) $H = \{n | n \in \mathbb{Z} \text{ and } |n| \geq 1\}, G = \mathbb{Z}$
   (d) $H = \{(m,n) | m,n \in \mathbb{Z} \text{ and } m+n \text{ is even}\}, G = \mathbb{Z} \times \mathbb{Z}$.
   (e) $H = \{1, -1, 0\}, G = \mathbb{Z}$.
   (f) $H = \{[0],[2],[4],[6]\}, G = \mathbb{Z}_8$.

2. In each case, determine whether H is a subgroup of the group $\mathbb{R}^* = (\mathbb{R} \setminus \{0\}, \cdot)$.
   (a) $H = \{1, -1\}$
   (b) H =the set of all positive real numbers.
   (c) H =the set of all positive integers.
   (d) $H = \{a + b\sqrt{3} \in \mathbb{R}^* | a, b \in \mathbb{Q}\}$

3. Let $GL_2(\mathbb{R})$ denote the group of all nonsingular 2×2 matrices with real entries. In each case, determine whether S is a subgroup of the group $GL_2(\mathbb{R})$
   (a) $S = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R}) | ad - bc = 1\right\}^2$

(b) $S = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \in GL(2,\mathbb{R}) | n \in \mathbb{Z} \right\}$

(c) $S = \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \in GL(2,\mathbb{R}) | b \text{ } is \text{ } nonzero \right\}$.

(d) $S = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in GL(2,\mathbb{R}) | ad > 0 \right\}$

(e) $S = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \in GL(2,\mathbb{R}) | a^2 + b^2 \neq 0 \right\}$

(f) $S = \left\{ \begin{bmatrix} a & 0 \\ b & 1 \end{bmatrix} \in GL(2,\mathbb{R}) | a \neq 0 \right\}$

(g) $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a,b,c,d \in \mathbb{Z} \right\}$is a group under addition of 2×2 matrices.

Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} | a,b,c,d \in \mathbb{Z}, a+b+c+d = 0 \right\}$. Prove H is a subgroup of G.

(h) Let $G = GL_2(\mathbb{R})$. Let $H = \{A \in G | \det A = 2^n \text{ } for \text{ } some \text{ } n \in \mathbb{Z}\}$. Prove that H is a subgroup of G.

(i) Let $G = GL_2(\mathbb{R})$ and $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} | a,b \in \mathbb{Z}, a \neq 0, b \neq 0 \right\}$. Prove or disprove: H is a subgroup of G.

(j) Let $G = GL_2(\mathbb{R}), H = \{A \in G \mid \det A \in Q\}$. prove or disprove: H is a subgroup of G.

(k) Show $H_3(\mathbb{R}) = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x,y,z \in \mathbb{R} \right\}$ is a subgroup of $GL_3(\mathbb{R})$ called Heisenberg Group.

4. Let $\mathbb{R}^*$ be the group of non-zero real numbers under multiplication and let $H = \{H \in \mathbb{R}^* | x^2 \in Q\}$. Prove that H is a subgroup of $\mathbb{R}^*$.

5. Show that the set $H = \{a + ib \in \mathbb{C}^* | a^2 + b^2 = 1\}$ is a subgroup of $(\mathbb{C}^*, \cdot)$ where $\cdot$ is the usual multiplication of complex numbers.
6. If G is a Abelian group, then prove that $H = \{a^2 | a \in G\}$ is a subgroup of G.
7. If G is a Abelian group, then prove that $H = \{a \in G | a^2 = e\}$ is a subgroup of G.
8. Let K be a subgroup of a group G and H be a subgroup of K.Is it true that H is a subgroup of G? Justify.
9. Let G be group and $a \in G$. Show that $H = \{a^{2n} : n \in Z\}$ is a subgroup of G.
10. Let G be a Abelian group. Prove that the set H of all elements of finite order in G is a subgroup of G.
11. Let G be a Abelian group. Prove that the subset $H = \{a \in G | o(a) divides \text{ } 10\}$ is a subgroup of G.
12. In the group S_3, Show that the subset $H = \{a \in S_3 | o(a) divides \text{ } 2\}$ is not a subgroup.
13. Let G be a group. Prove that if G has finitely many subgroups, then$G$ is finite.
14. Does there exist an infinite group with only a finite number of subgroups? If yes, give example.

15. Let G be an Abelian group and n be a fixed positive integer. Let $G^n = \{g^n | g \in G\}$. Prove that G^n is a subgroup of G.
16. Let H be a subgroup of $\mathbb{R}$ under addition. Let $K = \{2^a | a \in H\}$. Prove that K is a subgroup of $\mathbb{R}^*$ under multiplication.
17. G is a group of functions from $\mathbb{R}$ to $\mathbb{R}^*$ under point wise multiplication.
    Let $H = \{f \in G | f(1) = 1\}$. Prove that H is a subgroup of G.
18. Let $H = \{a + bi | a, b \in \mathbb{R}, ab \geq 0\}$. Prove or disprove: H is a subgroup of $\mathbb{C}$ under addition.

19. In the symmetric group S_3, show that $H = \{e, (23)\}$ and $K = \{e, (12)\}$ are subgroups but $H \cup K$ is not a subgroup of S_3.

Practical 2.4

Properties of Cyclic group

1. Suppose G is a cyclic group such that G has exactly three subgroups viz. $G, \{e\}$ and a subgroup pf order 5. Then the order of G is
(a) 5 (b) 10 (c) 25 (d) 125

2. The number of elements of order 5 in $\mathbb{Z}_{1000}$ is
(a) 1 (b) 4 (c) 5 (d) none of these

3. Let G be a cyclic group of infinite order. Then the number of elements of finite order in G is
(a) 1 (b) 2 (c) 5 (d) 0

4. The number of generators of an infinite cyclic group is
(a) 1 (b) 2 (c) 0 (d) infinite

5. The number of elements of order 5 in the cyclic group of order 25 is
(a) 1 (ii) 2 (iii) 5 (iv) none of these

6. Let G be a group and $a \in G$. If $0(a) = 24$ then $\langle a^{21} \rangle \cap \langle a^{10} \rangle$ has order
(a) 1 (b) 2 (c) 3 (d) 4

7. The number of subgroups of $(\mathbb{Z}_{20}, +)$ is
(a) 6 (b) 5 (c) 3 (d) 2

8. G is a cyclic group having exactly 3 subgroups namely G itself, $\{e\}$ and a subgroup of order 7. Then order of G is
(a) 14 (b) 49 (c) 7 p where p is any prime (d) cannot say

9. The total number of elements of order 8 in $\left(\mathbb{Z}_{80,00,000,}, +\right)$ is
(a) 8 (b) 10,00,000 (c) 4 (d) None of the above

10. Let G be a group and $a \in G$. If $0(a) = 0(a^2)$, then $0(a)$ is
(a) even (b) odd (c) a prime (d) None of the above

11. Let G be an infinite cyclic group generated by α. The generators of $\langle a^3 \rangle$ are
(a) a^3, a^2 (b) a^3, a^{-3} (c) a^3, a^m where $(m, 3) = 1$ (d) a^3, a^p where p is prime

12. Consider the subgroup H generated by $\overline{3}$ of the group $(\mathbb{Z}_{15}, +)$, then the number of generators of H is
(a) 4 (b) 2 (c) 3 (d) 4

13. Let $G = \{\overline{4}, \overline{8}, \overline{12}, \overline{16}\}$ under multiplication modulo 20. Then
(a) G is cyclic and $G = \langle \overline{4} \rangle$ (b) G is not cyclic
(c) G is cyclic and $G = \langle \overline{8} \rangle$ (d) None of the above

14. Let $G = \langle a \rangle$ and order of G be 40. Then the elements of order 10 in G are
    (a) a^4, a^8, a^{12}, a^{16} (b) $a^4, a^{12}, a^{28}, a^{36}$ (c)$a^4, a^{16}, a^{24}, a^{32}$ (d) a^4, a^8, a^{24}, a^{28}

15. The generators of $20\mathbb{Z} \cap 30\mathbb{Z}$ are
    (a) $60, -60$ (b) $10, -10$ (c)$20, 30$ (d) None of the above

16. Let G be a group and $a \in G$ such that $O(\langle a^5 \rangle) = 12 = O(\langle a^4 \rangle)$. Then possible orders of 'a' are
    (a) 20 (b) 12 or 60 (c) 24 (d) None of these

17. Let G be a cyclic group of order n generated by 'a' then $\langle a^r \rangle = \langle a^s \rangle$ implies
    (a) $(r, s) = 1$(b) $s = (n, r)$ (c)$(n, r) = (n, s)$ (d) $r/(n, s)$

18. If G is a cyclic group of order 11, then number of generators of G is
    (a) 2 (b) 10 (c) 11 (d) None of the above

19. Let $n =$ Number of elements of order 4 in $(\mathbb{Z}_{4k}, +)$ and $m =$Number of elements of order 4 in $(\mathbb{Z}_{8k}, +)$ (k is a positive number). Then
    (a) $m = 2n$ (b) $m = n^2$ (c) $m < n$ (d) $m = n$

20. Let H be the subgroup of $(\mathbb{Z}_{156}, +)$ generated by the set $\{\overline{56}, \overline{84}\}$. Then H is generated by
    (a) $\overline{152}$ (b) $\overline{153}$ (c)$\overline{154}$ (d) None of the above

21. Every proper subgroup of S_3 is
    (a) cyclic (b) non-abelian (c) of order 3 (d) of order 2

22. The number of elements of order 5 in $\mathbb{Z}_{20}$ is
    (a) 1 (b) 2 (c) 4 (d) 5

23. The number of elements of order 10 in $\mathbb{Z}_{10}$ is
    (a) 4 (b) 10 (c) 5 (d) 0

24. Let G be a group and $a \in G$. If $o(a) = 20$, then $o(a^4)$ is
    (a) 15 (b) 5 (c) 12 (d) 20

25. If $H_1 = < \overline{20} >$ and $H_2 = < \overline{15} >$ in $\mathbb{Z}_{30}$under addition modulo 30, then
    (a) $|H_1| = 10$ and $|H_2| = 15$ (b) $|H_1| = 3$ and $|H_2| = 2$
    (c) $|H_1| = 5$ and $|H_2| = 3$ (d) None of these.

26. If $H_1 = < \overline{3} >$ and $H_2 = < \overline{15} >$ in $\mathbb{Z}_{18}$under addition modulo 18, then
    (a) $|H_1| = 6$ and $|H_2| = 6$ (b) $|H_1| = 6$ and $|H_2| = 5$
    (c) $|H_1| = 15$ and $|H_2| = 3$ (d) None of these.

27. Suppose G is a cyclic group and $o(G) = 24$. If for some $a \in G$ $a^8 \neq e$ and $a^{12} \neq e$ then
    (a) $G = < a >$ (b) $G = < a^4 >$
    (c) $< a >$ is a proper subgroup of G. (d) None of these.

28. If $m, n \in \mathbb{Z}$, then the generator of $< m > \cap < n >$ is

    (a) mn (b) $\gcd(m\,, n\,)$ (c) $\text{lcm}(m\,, n)$ (d) None of these.

29. Let p be prime. If a group G has more than $p - 1$ elements of order p, then

    (a) G is cyclic (b) G is not cyclic
    (c) G has a unique subgroup of order p. (d) None of these.

Descriptive Questions

1) Show that the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \cdot)$ are not cyclic where $\mathbb{Q}^* = \mathbb{Q} - \{0\}$ and $+, \cdot$ are usual addition, multiplication respectively.

2) List (i) all generators (ii) all subgroups of the cyclic group $(\mathbb{Z}_{15}, +)$.

3) List (i) all generators (ii) all subgroups of the cyclic group $\langle a \rangle$ of order 16.

4) Let $U(n) = \{\overline{x} \mid x \in \mathbb{N}, (x, n) = 1, 1 \le x \le n\}$ under multiplication modulo n.
   Determine which of the following groups are cyclic. Justify your answer.
   (i) $U(4)$ (ii) $U(5)$ (iii) $U(7)$ (iv) $U(8)$
5) Show that the group $U(2^n)$ is not cyclic if $n > 2$.

6) Prove that $H = \left\{ \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} | n \in \mathbb{Z} \right\}$ is a cyclic subgroup of $GL_2(\mathbb{R})$.
7) Let G be a group and $a, b \in G$. If $O(a) = 12, O(b) = 22$, and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$ then show that $a^6 = b^{11}$.
8) Let G be a cyclic group. If $6 | O(G)$and a is an element of order 6 in G then list all elements of order 6 in G and justify your answer.

9) Let G be a cyclic group of order 18 generated by 'a'. List
   (i) all subgroups of G (ii) all generators of each subgroup of G.

10) Let G be a cyclic group of order 144 generated by a. State orders of the following elements $a^5, a^8, a^{13}, a^{20}, a^{21}, a^{95}, a^{101}$. (Mention the result used)

11) Prove or disprove: In a cyclic group G generated by 'a'
    $O(b^2) = O(c^2) \Rightarrow O(b) = O(c)$where $b, c \in G$.
12) Show that the 7^{th} roots of unity form a cyclic group. Find all generators of this group.
13) Show that the cyclic group $(\mathbb{Z}, +)$ has only two generators.

14) Show that the every positive integer n, the nth roots of unity from a cyclic group.
15) Show that $(\mathbb{Q}^{+},\cdot),(\mathbb{Q}^{*},\cdot),(\mathbb{R}^{+},\cdot),(\mathbb{R}^{*},\cdot),(\mathbb{C}^{*},\cdot)$are not cyclic group.
16) Let $G=\langle a\rangle$ be a cyclic group pf order 20. Find all distinct elements of the subgroups
 (i) $\langle a^4\rangle$ (ii) $\langle a^7\rangle$.
17) Let G be a cyclic group of order 42. Find the number of elements of order 6 and the number of elements of order 7 in G.
18) Prove or disprove that every non Abelian group has a nontrivial cyclic subgroup.
19) Prove or disprove: $G=\{(1,1),(1,-1),(-1,1),(-1,-1)\}$ is a group under the operation $(a,b)(c,d)=(ac,bd)$ but not a cyclic group.
20) Let a and b be elements of a group. If $0(a)=10$ and $0(b)=21$. Show that $\langle a\rangle\cap\langle b\rangle=\{e\}$.

Practical No : 2.5
Cosets and Lagrange's Theorem
Objective Questions

(1) Let H be a subgroup of G and $a \in G$. $aH = H$ if and only if

(a) $a \notin H$ (b) $a \in H$

(c) $a^{-1} \notin H$ (d) None of these

(2) Let H be a subgroup of G and $a, b \in G$. $aH = bH$ if and only if

(a) $a \in H$ (b) $ab \in H$

(c) $a^{-1}b \in H$ (d) None of these

(3) Let H be a subgroup of G and $a, b \in G$. If $aH \neq bH$ then

(a) $aH \cap bH = \phi$ (b) $aH \cap bH \neq \phi$

(c) $aH \subset bH$ (d) None of these

(4) Let H be a subgroup of G . Which of the following statements are true

(i) $a \notin aH$

(ii) $|aH| = |bH| \quad \forall a, b \in G$

(iii) $aH = Ha$ if and only if $H = aHa^{-1}$

(a) (i) and (ii) (b) (ii) and (iii)

(c) (i) and (iii) (d) Only (ii)

(5) Let $G = \mathbb{Z}$ and $H = 5\mathbb{Z}$. Then the following pair of left cosets are not equal. Then

(a) 11 + $5\mathbb{Z}$ and -39 + $5\mathbb{Z}$. (b) 11 + $5\mathbb{Z}$ and -25 + $5\mathbb{Z}$

(c) 11 + $5\mathbb{Z}$ and -34 + $5\mathbb{Z}$. (d) None of these.

(6) Let H be a subgroup of G and $a \in G$. aH is a subgroup of G if and only if

(a) $a \notin H$ (b) $a \in H$

(c) $a^{-1} \notin H$ (d) None of these

(7) If $G = \mathbb{Z}$ with addition and $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$ then which of the following is true

(a) $11 + H = 17 + H$ (b) $11 + H = 17 + H$

(c) $7 + H = 23 + H$ (d) $H = 2 + H$

(8) The left cosets of $H = \{1,11\}$ in $U(30)$ are

(a) $7 + H, 13 + H, 19 + H$ (b) $7 + H, 13 + H, 23 + H$

(c) $H, 1 + H, 29 + H$ (d) None of these

(9) Suppose a has order 20, then number of cosets of $< a^5 >$in $< a >$ is

(a) 3 (b) 5

(c) 4 (d) None of these

(10) Let $G = \mathbb{C}^*$ under multiplication and $H = \{z \in \mathbb{C}^* : |z| = 1\}$, then cosets of H in G are

(a) $\{z \in \mathbb{C}^* : |z| = k\}\ \forall k \in \mathbb{R}^+$ (b) $\{z \in \mathbb{C}^* : |z.w| = 1\}\ \forall w \in \mathbb{C}^*$

(c) $\{z \in \mathbb{C}^* : |z + w| = 1\}\ \forall w \in \mathbb{C}^*$ (d) None of these

(11) Let K be a proper subgroup of H and H be proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

(a) 84 , 210 (b) Only 84

(c) Only 210 (d) None of these

(12) If $G = \mathbb{Z}_{17}$ then how many solutions does $x^{16} = \overline{1}$ has in G

(a) 16 (b) 15

(c) 17 (d) None of these

(13) If $G = GL(2, \mathbb{R})$ and $H = SL(2, \mathbb{R})$ then for any matrix A in G, the coset AH is

(a) The set of all 2 X 2 matrices with the same determinant as A

(b) The set of all 2 X 2 matrices with determinant equal to 1

(c) The set of all 2 X 2 matrices with determinant equal to -1

(d) None of these

(14) If G is a non-abelian group of order 10 then G has how many elements of order 2

(a) 5 (b) 6

(c) 4 (d) None of these

(15) Order of $U(n)$ for $n > 2$ is

(a) Even (b) Odd

(c) n-1 (d) None of these

(16) H and K be a subgroups of a group G. If $|H| = 12$ and $|K| = 35$. Find $|H \cap K|$

(a) 2 (b) 1

(c) 3 (d) None of these

(17) Let G be a group of order 25. Which of the following is true

(a) G is cyclic (b) $g^5 = e$ for all g in G

(c) G is cyclic or $g^5 = e$ for all g in G (d) None of these

(18) Number of cosets of $\mathbb{Z}$ in $\mathbb{Q}$ and $\left|\frac{p}{q} + \mathbb{Z}\right|$ is

(a) Finite, infinite (b) Infinite, infinite

(c) Infinite, finite (d) None of these

(19) Let $|G| = 8$ then

(a) G must have an element of order 2

(b) G must have an element of order 4

(c) G may not have an element of order 2

(d) None of these

(20) Which of the following is true

(a) Every odd ordered subgroup of D_n is cyclic

(b) Every subgroup of D_n is cyclic

(c) Only one subgroup of D_n is cyclic

(d) None of these

(21) Let $G = S_3$ and consider the subgroup $H = \{I, (12)\}$ of G. Then

(a) Every left coset of H in G is also a right coset.

(b) $[G: H] = 3$ and two left cosets of H in G are also right cosets.

(c) No left coset except H itself is a right coset of H in G.

(d) None of the above.

(22) Suppose G has subgroups of order 45 and 75. If $|G| < 400$, then $|G|$is

(a) 90 (b) 150

(c) 225 (d) None of this

(23) Let G be an abelian group of order 12. If $a, b \in G$ and order of a is 4 and order of b is 6 then order of ab is

(a) 2 (b) 4

(c) 6 (d) 12

Descriptive Questions

(1) Let $\mathrm{H} = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$. Find all the left cosets of H in Z.

(2) Find all distinct left cosets of the subgroup H in the group G.

(a) $\mathrm{H} = \{1, -1\}, \mathrm{G} = (\mathbb{R} \backslash \{0\}, \cdot)$.

(b) $\mathrm{H} = 7\mathbb{Z},\ \mathrm{G} = \mathbb{Z}$

(c) $\mathrm{H} = \{\mathrm{e}, (23)\}, \mathrm{G} = \mathrm{S}_3$.

(d) $\mathrm{H} = \{\mathrm{e}, (1\ 2\ 3), (1\ 3\ 2)\}, \mathrm{G} = \mathrm{S}_3$.

(3) Let H = {(1),(12)(34),(13)(24),(14)(23)}. Find the left cosets of H in A_4.

(4) Find all of the left cosets of {1,11} in U(30).

(5) In the multiplicative group $\mathbb{C}^*$, describe geometrically the coset $(2 + 3\mathrm{i})\mathrm{H}$ of the subgroup $\mathrm{H} = \{\mathrm{z} \in \mathbb{C}^* : |\mathrm{z}| = 1\}$.

(6) Let $G = \mathbb{R} \times \mathbb{R}$ be the group under binary operation $*$ defined by $(a, b) * (c, d) = (a + c, b + d)$. Let $H = \{(a, 5a) : a \in \mathbb{R}\}$. Show that H is a subgroup of G. Describe the left cosets of H in G. Describe this subgroup and its cosets geometrically.

(7) Show that the set of all left cosets of the subgroup $8\mathbb{Z}$ in the group $(\mathbb{Z}, +)$ is given by $\mathcal{L} = \{x + 8\mathbb{Z} | x = 0,1,2, \ldots, 7\}$.

(8) Determine whether or not the following cosets of the subgroup $H = 5\mathbb{Z}$ in the group $(\mathbb{Z}, +)$ are equal:

(i) $-1 + H$ and $5 + H$

(ii) $3 + H$ and $2 + H$

(9) Show that the set $\mathcal{L}$ of all left cosets of $\mathbb{Z}$ in the additive group $(\mathbb{R}, +)$ of all real numbers is given by $\mathcal{L} = \{x + \mathbb{Z} | 0 \leq x < 1\}$.

(10) Let G be a group of order pq where p and q are prime integers. Show that every subgroup $H \neq G$ is a cyclic subgroup.

(11) Find all subgroups of Klien's four group.

(12) Prove that every proper subgroup of S_3 is cyclic.

(13) Prove that every group of order 4 is an Abelian group.

(14) Prove that every group of order 49 contains a subgroup of order 7.

(15) If a group G has only two subgroups, then prove that G is a cyclic group.

(16) Let G be a group. If a subset A is a left coset of some subgroup of G, then show that A is a right coset of some subgroup of G.

(17) Let A and B be two subgroups of a group G. If $|A| = p$, a prime integer, then show that either $A \cap B = \{e\}$ or $A \subseteq B$.

(18) Let G be a group and H and K be subgroups of G. Show that $(H \cap K)x = Hx \cap Kx$ for all $x \in G$.

(19) Let G be a group and H and K be subgroups of G. Let $a, b \in G$. Show that either $Ha \cap Kb = \emptyset$ or $Ha \cap Kb = (H \cap K)c$ for some $c \in G$.

(20) Let G be a group and H and K be subgroups of G of finite indices. Show that $H \cap K$ is of finite index.

(21) Give an example of a group G and a subgroup H of G, such that $aH = bH$ but $Ha \neq Hb$ for some $a, b \in G$.

Practical No : 2.6
Group Homomorphism, Isomorphism and Automorphism
Objective Questions

(1) Let $\varphi: C^* \to C^*$ given by $\varphi(x) = x^4$ be a homomorphism then $ker\varphi =$

(a) $\{1, -1\}$ (b) $\{1, -1, i, -i\}$

(c) $\{i, -i\}$ (d) None of these

(2) Let $\varphi: \mathbb{Z}_{12} \to Z_{12}$ given by $\varphi(x) = 3x$ be a homomorphism then $ker\varphi =$

(a) $\{\bar{0}, \bar{1}\}$ (b) $\{\bar{0}, \bar{4}, \bar{8}\}$

(c) $\{\bar{0}, \bar{4}\}$ (d) None of these

(3) Number of homomorhisms from $\mathbb{Z}_{12}$ to $\mathbb{Z}_{30}$ is

(a) 6 (b) 7

(c) 8 (d) None of these

(4) Let G be a cyclic group of order 7 and $\varphi: G \to G$ given by $\varphi(x) = x^4$ then

(a) φ is not a group homomorphism

(b) φ is a group homomorphism which is not one-one

(c) φ is a group homomorphism which is not onto

(d) None of these

(5) Suppose $\varphi: \mathbb{Z}_{30} \to Z_{30}$ is a group homomorphism and $ker\varphi = \{\bar{0}, \overline{10}, \overline{20}\}$. If $\varphi(\overline{23}) = \bar{9}$then the elements that map to $\bar{9}$ are

(a) $\bar{3}, \overline{13}, \overline{23}$ (b) $\bar{3}, \overline{23}$

(c) $\overline{13}, \overline{23}$ (d) None of these

(6) Let G be a subgroup of some dihedral group. Define $\varphi: G \to \{1, -1\}$

$\varphi(x) = \begin{cases} 1 \text{ if x is a rotation} \\ -1 \text{ if x is a reflection} \end{cases}$ then $ker\varphi =$

(a) A subgroup containing only reflections.

(b) A subgroup containing only reflections and rotations.

(c) A subgroup containing only rotations.

(d) None of these

(7) How many homomorphisms are there from $\mathbb{Z}_{20}$ onto $\mathbb{Z}_{10}$?

(a) 4 (b) 5

(c) 10 (d) None of these

(8) Which of the following statements is true.

(a) $(\mathbb{Z}_4, +)$ and V_4(Klein's 4 group) are isomorphic. (b) $(\mathbb{Z}, +)$and μ_4 (The group of fourth roots of unity) are isomorphic.

(c) $V_4$and $\mu_4$are isomorphic (d) None of these

(9) Let Q_8 be quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$ $where\ i^2 = j^2 = -1, ij = k = -ji$. Consider the map $\varphi: Q_8 \to \mathbb{Z}_2$ defined by $\varphi(i) = \bar{0}, \varphi(j) = \bar{1}$ then

(a) φ is not a group homomorphism

(b) φ is a group homomorphism and $ker\varphi = \{1, i\}$

(c) φ is a group homomorphism and $ker\varphi = \{-i, i\}$

(d) None of these

(10) Suppose for each prime p, $\mathbb{Z}_p$ is the homomorphic image of a group G. What can we say about $|G|$?

(a) finite (b) infinite

(c) prime (d) None of these

(11) Let G be an abelian group which has no element of order 2 and $\varphi: G \to G$ given by $\varphi(x) = x^2$, then

(a) φ is an automorphism of G.

(b) φ is a group homomorphism which may not be one-one.

(c) φ is an automorphism of G if G is finite.

(d) φ is not a group homomorphism.

(12) Let $\varphi: \mathbb{Z}_{12} \to Z_{10}$ given by $\varphi(x) = 3x$

(a) φ is not a group homomorphism

(b) φ is a group homomorphism which is not one-one

(c) φ is a group homomorphism which is not onto

(d) None of these

(13) Consider the group G, where $G = \left\{\begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \epsilon\mathbb{R}\ , a \neq 0\right\}$ under multiplication of 2x2 matrices. then

(a) G is isomorphic to $(\mathbb{R}\ , +)$.

(b) G is isomorphic to $(\mathbb{R}^*\ , +)$

(c) G is isomorphic to $(SL_2(\mathbb{R})\ , +)$

(d) None of these

(14) Let U(16) denote the group of prime residue classes modulo 16 under multiplication. Which of the following statements are false?

(a) $\varphi_1: U(16) \to U(16)$defined by $\varphi_1(x) = x^3$ is a group automorphism.

(b) $\varphi_2: U(16) \to U(16)$defined by $\varphi_2(x) = x^5$ is a group automorphism.

(c) $\varphi_3: U(16) \to U(16)$defined by $\varphi_3(x) = x^9$ is a group automorphism.

(d) $\varphi_4: U(16) \to U(16)$defined by $\varphi_1(x) = x^4$ is a group automorphism.

(15) Let m and n be integers. Then

(a) The groups $(m\mathbb{Z}, +)$ and $(n\mathbb{Z}, +)$ are isomorphic.

(b) The groups $(m\mathbb{Z}, +)$ and $(n\mathbb{Z}, +)$ are isomorphic if and only if $m = -n$.

(c) The groups $(m\mathbb{Z}, +)$ and $(n\mathbb{Z}, +)$ are not isomorphic if $m \neq n$.

(d) The groups $(m\mathbb{Z}, +)$ and $(n\mathbb{Z}, +)$ are isomorphic for all non-zero integers m and n.

(16) The map $f: GL_2(\mathbb{R}) \to GL_2(\mathbb{R})$ defined by $f(A) = (A^t)^{-1}$ is

(a) not a group homomorphism.

(b) group homomorphism and $kerf = SL_2(\mathbb{R})$.

(c) group homomorphism and $kerf = O_2(\mathbb{R})$.

(d) a group automorphism

(17) The number of group homomorphism from $S_3\ to\ (\mathbb{Z}_3, +)$ is

(a) 1 (b) 0

(c) 3 (d) 2

(18) The number of group automorphism of V_4 (Klien's four group) is

(a) 4 (b) 2

(c) 3 (d) 6

(19) Let G be an abelian group of order n. The map $\varphi: G \to G$ given by $\varphi(x) = x^m$ where m is a positive integer is

(a) a group homomorphism if and only if m,n are relatively prime.

(b) a group homomorphism and $kerf = \{x \epsilon G : x^d = e\}$ where $d = gdc(m,n)$

(c) a group homomorphism and ker f has order m.

(d) None of these.

(20) The number of onto homomorphisms from S_3 to V_4 are

(a) 1 (b) 0

(c) 2 (d) 3

Descriptive Questions

(1) Show that the following are group homomorphism. In each case find Kernel and check if it is an isomorphism

(i) $f: GL_2(\mathbb{R}) \to (\mathbb{R}^*,.)$defined by $f(A) = \det A$.

(ii) $f : U(5) \to U(5)$defined by $f(x) = x^2$.

(iii) $f : (\mathbb{R}, +) \to (\mathbb{R}^2,.)$defined by $f(x) = e^2$.

(iv) $f : (\mathbb{C}^*,.) \to (\mathbb{R}^+,.)$ defined by $f(z) = |z|$.

(2) Let $f: S_n \to \{1,-1\}$ be given by $f(x) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation} \\ -1 & \text{if } \sigma \text{ is an odd permutation} \end{cases}$ is a homomorphism where {1,-1} is group under multiplication. Also find Kerf.

(3) Let G be a group $a \in G$ be non-identity. Show that $f: G \to G$ $given\ by$

$f(x) = a^{-1}x\,a$ is an automorphism

(4) Check whether the following pairs of groups are isomorphic or not

(i) $(\mathbb{Q}, +)\, and\ (\mathbb{Q}^+,.)$

(ii) $(\mathbb{Q}, +)\, and\ (\mathbb{Q}^*,.)$

(iii) $(\mathbb{Q}, +) and\ (\mathbb{Z}, +)$

(iv) $(\mathbb{Q}^*, .) and\ (\mathbb{Z}, +)$

(5) (i) Show that the group U(8) is not isomorphic to the group U(10) but U(8) is isomorphic U(12).

(6) Prove or disprove that U(20) and U(24) are isomorphic groups.

(7) Show that the group $G = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ under multiplication of 2x2 matrices is isomorphic to Klien's four group where $V_4 = \{e, a, b, c\}, a^2 = b^2 = e, ab = ba = c.$

(8) How many homomorphism are there from $\mathbb{Z}_{20}\ to\ \mathbb{Z}_8$? How are of them are onto homomorphisms?

(9) Find a homomorphism $\varphi: U(30) \to U(30)$ such that $ker\varphi = \{\overline{1}, \overline{11}\}$ and $\varphi(\overline{7}) = \overline{7}$.

(10) Suppose $\varphi: U(30) \to U(30)$ is a group homomorphism and $ker\varphi = \{\overline{1}, \overline{11}\}$. If $\varphi(\overline{7}) = \overline{7}$ then find all other elements of U(30) that map to $\overline{7}$.

(11) Show that G is Abelian if and only if $f: G \to G$ given by $f(x) = x^2$ is a homomorphism.

(12) Show that G is Abelian if and only if $f: G \to G$ given by $f(x) = x^{-1}$ is a homomorphism.

(13) Show that there does not exit any non-zero homomorphism of the group S_3 to the group $\mathbb{Z}_3$ under addition modulo 3.

(14) Let $f, g : G \to G'$ be a group homomorphism and $H = \{x \in G : f(x) = g(x)\}$. Prove or disprove H is a subgroup of G.

(15) Show that the mapping $\varphi: (\mathbb{C}, +) \to (\mathbb{C}, +)\ given\ by\ \varphi(a + bi) = a - bi$ is an automorphism.

(16) Let $G = \{a + b\sqrt{2} : a, b \text{ are rationals}\}$ and $H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \text{ are rationals} \right\}$. Show that G and H are isomorphic under addition.

Does your isomorphism preserve multiplication as well as addition?

(17) Suppose that G is finite group and that $\mathbb{Z}_{10}$ is a homomorphic image of G. What can be said about o(G)?

(18) Suppose that $\mathbb{Z}_{10}$ and $\mathbb{Z}_{15}$ are both homomorphism images of a finite group G. What can be said about o(G)?

Practical 2.7 Unitwise Theory Questions

Unit I : Groups and Subgroups

(1) Let G be a group. Then prove that

i) Identity element of G is unique.

ii) Inverse of an element in G is unique.

iii) Cancellation laws holds in G; that is, $ab = ac \Longrightarrow b = c and ba = ca \Longrightarrow b = c$ for all $a, b, c \in G$.

iv) $(a^{-1})^{-1} = a$ for all $a \in G$.

v) $(ab)^{-1} = b^{-1}a^{-1}$ for all $a, b \in G$.

vi) $(aba^{-1})^n = ab^n a^{-1}$ for all $a, b \in G$ and $n \in \mathbb{Z}$.

(2) Define Abelian group. Show that a group G is Abelian in each of the following cases,

i) $(ab)^2 = a^2b^2$ for all $a, b \in G$.

ii) $a^2 = e$ or $a^{-1} = a$ for all $a \in G$, where e is identity of G.

iii) $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Show that converse is also true.

(3) An element $\bar{a} \in \mathbb{Z}_n$ has multiplicative inverse if and only if $(a , n) = 1$, where (a , n) denotes G.C.D of a and n.

(4) $(\mathbb{Z}_n^* , \cdot)$ is a group if and only if n is prime.

(5) Define order of an element in a group G. For $a, b \in G$, prove that

i) $o(a) = o(a^{-1})$ ii) $o(a) = o(bab^{-1})$ iii) $o(ab) = o(ba)$

(6) Let G be a group and $a \in G$ with $o(a) = n$ then $a^m = e$ if and only if $n \mid m$.

(7) Let G be a group and $a \in G$. If $o(a) = n$ then $o(a^m) = \frac{n}{(m,\ n)}$, where (m , n) denotes G.C.D of m and n.

(8) Let G be a group and $a \in G$. If $o(a) = mn$ then $o(a^n) = m$

(9) Let G be a group , $a, b \in G$ such that $o(a) = m$, $o(b) = n$ and $ab = ba$. If $(m , n) = 1$ then $o(ab) = mn$

(10) If G is a group of even order then show that G has an element of order 2.

(11) Define Subgroup of a group. State and prove necessary and sufficient condition for a non−empty set to be a subgroup.

(12) Let H be a finite subset of a group $(G ,*)$ then H is subgroup of G if and only if $a * b \in H$, $\forall\, a, b \in H$.

(13) Let H and K be subgroups of a group G then show that $H \cap K$ is also a subgroup of G. Is $H \cup K$ a subgroup of G? Justify. Further show that intersection of family of subgroups of $G$is

again a subgroup of G.

(14) Let H and K be subgroups of a group G then $H \cup K$ is also a subgroup of G if and only if either $H \subseteq K \ or \ K \subseteq H$.

(15) Define Center $Z(G)$ of a group G. Show that $Z(G)$ is a subgroup of G. Further if G is Abelian group then $Z(G) = G$.

(16) If H and K be subgroups of a group G then show that HK is a subgroup of G if and only if $HK = KH$.

Unit II :Cyclic Groups and Cyclic Subgroups

(1) Define Cyclic group and show that a cyclic group is Abelian.

(2) Let G be a finite group then $'a'$ is generator of G if and only if $o(a) = o(G)$.

(3) If G is infinite cyclic group then show that G has exactly two generators.

or

$a \ and \ a^{-1}$ are the only generators of an infinite cyclic group.

(4) Let G be a finite cyclic group of order n generated by $'a'$ then a^m is also generator of G if and only if $(m \, , \, n) = 1$.

(5) Show that every subgroup of a cyclic group is cyclic.

(6) Let G be an infinite cyclic group generated by $'a'$. Show that

i) Every non$-$trivial subgroup of G is infinite cyclic.

ii) G has infinitely many distinct subgroups.

(7) Let G be a finite cyclic group of order n then G has unique subgroup of order d for each divisor d of n.

(8) Show that every group of prime order p is cyclic. Further show that it has $p - 1$ generators.

(9) Let G be a cyclic group of order p^n and $H \ and \ K$ are subgroups of G then prove that either $H \subseteq K \ or \ K \subseteq H$.

Unit III :Lagrange's Theorem and Group Homomorphism

(1) Let H be a subgroup of G and $a, b \in G$ then

(i) $a \in aH$

(ii) $aH = bH \ or \ aH \cap bH = \emptyset$

(iii) $|aH| = |bH|$

(iv) $aH = Ha \ if \ and \ only \ H = aHa^{-1}$.

(2) Let H is a subgroup of a group G and $x \in G$ then $xH = H$ $(or\ Hx = H)$ if and only if $x \in H$.

(3) Let H is a subgroup of a group G and $x, y \in G$ then
i) $xH = yH$ if and only if $x^{-1}y \in H$.
ii) $Hx = Hy$ if and only if $xy^{-1} \in H$.

(4) Let G be a group and H is subgroup of G then show that aH is also subgroup of G if and only if $a \in H$.

(5) State and prove Lagrange's theorem for a finite group.

(6) If G has no non−trivial subgroups then show that G is cyclic group of prime order.

(7) Let G be a group of prime order p. If H and K are subgroups of G then show that either $H \cap K = \{e\}$ or $H = K$

(8) In a finite group, show that the order of each element of the group divides the order of the group.

(9) Show that a group of prime order is cyclic.

(10) Let G be a finite group and $a \in G$, then prove that $a^{|G|} = e$.

(11) State and prove Fermat's Little theorem.

(12) Define Kernel of a group homomorphism. If $f: G \longrightarrow G'$ is group homomorphism then show that $Kerf$ is subgroup of G. Further f is injective if and only if $kerf = \{e\}$.

(13) Let $f: G \longrightarrow G'$ is onto group homomorphism. Prove that
i) If H is subgroup of G then $f(H) = \{f(h) : h \in H\}$ is subgroup of G'.
ii) If H' is subgroup of G' then $f^{-1}(H') = \{a \in G : f(h) \in H'\}$ is subgroup of G' and

$$kerf \subseteq f^{-1}(H').$$

(14) Define automorphism of groups. Let $a \in G$, show that $f_a: G \longrightarrow G$ defined by $f_a(x) = axa^{-1}, \forall\, x \in G$ is an automorphism

(15) Let G be a group. Show that $f: G \longrightarrow G$ defined by $f(x) = x^{-1}, \forall\, x \in G$ is an automorphism if and only if G is Abelian.

(16) Let G be an Abelian group of order n and $(m, n) = 1$, $m \in \mathbb{Z}$ then show that $f: G \longrightarrow G$ defined by $f(x) = x^m, \forall\, x \in G$ is an automorphism .